The Reimann Hypothesis

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October 26, 2020

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln \ln n$ where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann Hypothesis is true. We prove the Robin's inequality is true for every natural number $n > 5040$. In this way, we demonstrate the Riemann Hypothesis is true.

1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [2]. It is of great interest in number theory because it implies results about the distribution of prime numbers [2]. It was proposed by Bernhard Riemann (1859), after whom it is named [2]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [2].

The sum-of-divisors function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [6]. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality

$$\sigma(n) < e^{\gamma} \times n \times \ln \ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [4]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true [4]. Using this inequality, we show the Riemann Hypothesis is true.

2010 *Mathematics Subject Classification.* Primary 11M26; Secondary 11A41.
Key words and phrases. number theory, inequality, sum-of-divisors function, prime.

2. Results

Theorem 2.1. *Given a natural number*

$$n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$$

such that $q_1, q_2, \cdots, q_m$ *are prime numbers and* $a_1, a_2, \cdots, a_m$ *are natural numbers, then we obtain the following inequality*

$$\frac{\sigma(n)}{n} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof. From the article reference [1], we know

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}. \tag{2.1}$$

We can easily prove

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} = \prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

However, we know

$$\prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}}$$

where q_j is the j^{th} prime number and

$$\prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [6]. Consequently, we obtain

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}$$

and thus,

$$\frac{\sigma(n)}{n} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

□

Theorem 2.2. *The Robin's inequality is true for every natural number* $n > 5040$ *when the greatest prime divisor* q_m *of* n *complies with* $q_m \leq 5$.

Proof. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers, we need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln \ln n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \ln \ln n$$

according to the inequality (2.1). Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln\ln(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \ln\ln(5040) < e^{\gamma} \times \ln\ln n$$

and therefore, the proof is completed. □

Definition 2.3. We recall that an integer n is said to be squarefree if for every prime divisor q of n we have $q^2 \nmid n$, where $q^2 \nmid n$ means that q^2 does not divide n [1].

Theorem 2.4. *The Robin's inequality is true for every natural number $n > 5040$ when $(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$ such that n' is the squarefree kernel of n.*

Proof. We will check the Robin's inequality for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers. We need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \ln\ln n$$

according to the Theorem 2.1. From any squarefree number n', we obtain

$$\sigma(n') = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{2.2}$$

when $n' = q_1 \times q_2 \times \cdots \times q_m$ [1]. Using the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \ln\ln n$$

where $n' = q_1 \times \cdots \times q_m$ is the squarefree kernel of n [1]. However, the Robin's inequality has been proved for all the squarefree integers $n' \notin \{2, 3, 5, 6, 10, 30\}$ [1]. In addition, due to the Theorem 2.2, the Robin's inequality is true for every natural number $n > 5040$ when $n' \in \{2, 3, 5, 6, 10, 30\}$, where n' is the squarefree kernel of n. In this way, we have

$$\frac{\sigma(n')}{n'} < e^{\gamma} \times \ln\ln n'$$

and therefore, it is enough to prove

$$\frac{\pi^2}{6} \times e^{\gamma} \times \ln\ln n' \leq e^{\gamma} \times \ln\ln n$$

which is the same as

$$\frac{\pi^2}{6} \times \ln\ln n' \leq \ln\ln n$$

and

$$\ln(\ln n')^{\frac{\pi^2}{6}} \leq \ln\ln n$$

that is true when

$$(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$$

and thus, the proof is completed. □

The following Lemma is a very helpful inequality:

Lemma 2.5. *We have*

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}$$

where $y = 1 - x$.

Proof. We know $1 + x \leq e^x$ [3]. Therefore,

$$\frac{x}{1-x} \leq \frac{e^{x-1}}{1-x} = \frac{1}{(1-x) \times e^{1-x}} = \frac{1}{y \times e^y}.$$

However, from the article reference [3],

$$y \times e^y \geq y + y^2 + \frac{y^3}{2}$$

and this can be transformed into

$$\frac{1}{y \times e^y} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

Consequently, we show

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

□

Theorem 2.6. *The Robin's inequality is true for every natural number* $n > 5040$ *when the greatest prime divisor* q_m *of* n *complies with* $q_m \geq 7$.

Proof. We are going to prove this Theorem for every natural number $n > 5040$ using the following two possible cases under the assumption that the greatest prime divisor q_m of n complies with $q_m \geq 7$.

Case 1: $q_m^{e^\gamma} < \ln n$.

According to the Theorem 2.4, we know the Robin's inequality is true for every natural number $n > 5040$ when $(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$ such that n' is the squarefree kernel of n. In this way, we need to prove for the remaining case, that is when $(\ln n')^{\frac{\pi^2}{6}} > \ln n$, $q_m \geq 7$ and $q_m^{e^\gamma} < \ln n$. That would equivalent to

$$(\ln n')^{\frac{\pi^2}{6}} > q_m^{e^\gamma}$$

which is the same as

$$\ln n' > q_m^{\frac{6 \times e^\gamma}{\pi^2}}.$$

We denote by $\vartheta(x)$ the logarithm of the product of all primes lesser than or equal to x [5]. We know $\vartheta(q_m) \geq \ln n'$ and thus, we would have

$$\vartheta(q_m) > q_m^{\frac{6 \times e^\gamma}{\pi^2}}.$$

From the article reference [5], we have for $x > 0$

$$\vartheta(x) < 1.01624 \times x.$$

In this way, we obtain

$$1.01624 \times q_m > q_m^{\frac{6 \times e^\gamma}{\pi^2}}$$

and since we know $\frac{6\times e^{\gamma}}{\pi^2} > 1$, then we only need to prove

$$1.01624 > q_m^{\frac{6\times e^{\gamma}}{\pi^2}-1}.$$

However, we know

$$1.01624 < 7^{\frac{6\times e^{\gamma}}{\pi^2}-1} \leq q_m^{\frac{6\times e^{\gamma}}{\pi^2}-1}$$

and consequently, we obtain a contradiction just assuming that $(\ln n')^{\frac{\pi^2}{6}} > \ln n$ when $q_m \geq 7$ and $q_m^{e^{\gamma}} < \ln n$. Hence, this implies that necessarily $(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$ when $q_m \geq 7$ and $q_m^{e^{\gamma}} < \ln n$ and therefore, the Robin's inequality is true for this case when the greatest prime divisor q_m of n complies with $q_m \geq 7$.

Case 2: $q_m^{e^{\gamma}} \geq \ln n$.

Suppose that n is the smallest integer exceeding 5040 that does not satisfy the Robin's inequality, where n necessarily complies with this case. Let $n = r \times q_m$, where $q_m \geq 7$ denotes the largest prime factor of n. Under this assumption, we have

$$q_m \geq (\ln n)^{e^{-\gamma}}$$

should be true, which is equivalent to

$$\frac{1}{(\ln n)^{e^{-\gamma}}} \geq \frac{1}{q_m}.$$

In this way, the following inequality

$$\frac{\sigma(n)}{n} \geq e^{\gamma} \times \ln \ln n$$

should be true as well. From the article reference [1], we know

$$(1 + \frac{1}{q_m}) \times \frac{\sigma(r)}{r} \geq \frac{\sigma(q_m \times r)}{q_m \times r} \geq \frac{\sigma(n)}{n} \geq e^{\gamma} \times \ln \ln n.$$

Besides, this shows

$$(1 + \frac{1}{q_m}) \times e^{\gamma} \times \ln \ln r > e^{\gamma} \times \ln \ln n$$

should be true. Certainly, if n is the smallest counterexample of the Robin's inequality exceeding 5040, then the Robin's inequality is satisfied on r [1]. That is the same as

$$(1 + \frac{1}{q_m}) \times \ln \ln r > \ln \ln n.$$

We already obtain

$$\frac{1}{(\ln n)^{e^{-\gamma}}} \geq \frac{1}{q_m}$$

and thus,

$$(1 + \frac{1}{(\ln n)^{e^{-\gamma}}}) \times \ln \ln r > \ln \ln n$$

should be also true. We have

$$(1 + \frac{1}{(\ln n)^{e^{-\gamma}}}) \times \ln \ln r > \ln(\ln r + \ln q_m)$$

where we notice that $\ln(a + c) = \ln a + \ln(1 + \frac{c}{a})$. This follows

$$(1 + \frac{1}{(\ln n)^{e^{-\gamma}}}) \times \ln \ln r > \ln \ln r + \ln(1 + \frac{\ln q_m}{\ln r})$$

which is equal to

$$(1 + (\ln n)^{e^{-\gamma}}) \times \ln \ln r > (\ln n)^{e^{-\gamma}} \times \ln \ln r + (\ln n)^{e^{-\gamma}} \times \ln(1 + \frac{\ln q_m}{\ln r})$$

and thus,

$$\ln \ln r > (\ln n)^{e^{-\gamma}} \times \ln(1 + \frac{\ln q_m}{\ln r}).$$

This implies

$$\frac{\ln \ln r}{\ln(1 + \frac{\ln q_m}{\ln r})} =$$
$$\frac{\ln \ln r}{\ln \frac{\ln r + \ln q_m}{\ln r}} =$$
$$\frac{\ln \ln r}{\ln \frac{\ln n}{\ln r}} =$$
$$\frac{\ln \ln r}{\ln \ln n - \ln \ln r} =$$
$$\frac{\ln \ln r}{\ln \ln n \times (1 - \frac{\ln \ln r}{\ln \ln n})} =$$
$$\frac{\frac{\ln \ln r}{\ln \ln n}}{(1 - \frac{\ln \ln r}{\ln \ln n})} > (\ln n)^{e^{-\gamma}}$$

should be true. If we assume that $y = 1 - \frac{\ln \ln r}{\ln \ln n}$, then we analyze

$$\frac{1}{y + y^2 + \frac{y^3}{2}} \geq \frac{\frac{\ln \ln r}{\ln \ln n}}{(1 - \frac{\ln \ln r}{\ln \ln n})}$$

because of Lemma 2.5. As result, we have

$$\frac{1}{y + y^2 + \frac{y^3}{2}} > (\ln n)^{e^{-\gamma}}$$

and therefore,

$$1 > (\ln n)^{e^{-\gamma}} \times (y + y^2 + \frac{y^3}{2})$$

should be true. In addition, we already state

$$y = 1 - \frac{\ln \ln r}{\ln \ln n} = \frac{\ln \ln n - \ln \ln r}{\ln \ln n} = \frac{\ln(1 + \frac{\ln q_m}{\ln r})}{\ln \ln n}$$

where we have $\frac{x}{x+1} \leq \ln(1 + x)$ for $x > -1$ [3]. Hence, we note

$$\frac{\ln(1 + \frac{\ln q_m}{\ln r})}{\ln \ln n} \geq \frac{\frac{\ln q_m}{\ln r}}{(\frac{\ln q_m}{\ln r} + 1) \times \ln \ln n} = \frac{\ln q_m}{(\ln n) \times \ln \ln n}.$$

Moreover, we obtain

$$\frac{\ln q_m}{(\ln n) \times \ln \ln n} = \frac{1}{(\frac{\ln n}{\ln q_m}) \times \ln \ln n} > \frac{1}{\ln \ln n}$$

since $\frac{\ln n}{\ln q_m} > 1$. For that reason, we deduce

$$(y + y^2 + \frac{y^3}{2}) > ((\frac{1}{\ln \ln n}) + (\frac{1}{\ln \ln n})^2 + \frac{(\frac{1}{\ln \ln n})^3}{2})$$

and therefore, we finally obtain

$$1 > (\ln n)^{e^{-\gamma}} \times ((\frac{1}{\ln \ln n}) + (\frac{1}{\ln \ln n})^2 + \frac{(\frac{1}{\ln \ln n})^3}{2})$$

should be true too. However, we can see

$$\begin{aligned}1 &> (\ln n)^{e^{-\gamma}} \times ((\frac{1}{\ln \ln n}) + (\frac{1}{\ln \ln n})^2 + \frac{(\frac{1}{\ln \ln n})^3}{2}) \\ &> (\ln 5040)^{e^{-\gamma}} \times ((\frac{1}{\ln \ln 5040}) + (\frac{1}{\ln \ln 5040})^2 + \frac{(\frac{1}{\ln \ln 5040})^3}{2})\end{aligned}$$

is indeed false and as a consequence, we obtain a contradiction just assuming this case and the existence of a counterexample of the Robin's inequality exceeding 5040. In conclusion, the proof is completed. □

Theorem 2.7. *The Robin's inequality is true for every natural number $n > 5040$.*

Proof. This result is a consequence of the Theorems 2.2 and 2.6. □

Theorem 2.8. *The Riemann Hypothesis is true.*

Proof. If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann Hypothesis is true [4]. To sum up, this is true according to the Theorem 2.7. □

3. Conclusions

The practical uses of the Riemann Hypothesis include many propositions which are known true under the Riemann Hypothesis, and some of them can be shown equivalent to the Riemann Hypothesis [2]. Certainly, the Riemann Hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf Hypothesis, the large prime gap conjecture, etc [2]. In this way, this proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics in general [2].

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