Minimally Many-Valued Maximally Paraconsistent Minimal Unary Subclassical Expansions of LP

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MINIMALLY MANY-VALUED MAXIMALLY PARACONSISTENT MINIMAL UNARY SUBCLASSICAL EXPANSIONS OF *LP*

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Abstract. Here, for any $n > 2$, we propose a *minimally* n-valued (i.e., m-valued, for no $0 < m < n$) *maximally* paraconsistent (i.e., having no proper paraconsistent extension) subclassical (i.e., having a classical extension) expansion C_n of the *logic of paradox LP* by solely unary connectives, none of which can be eliminated with retaining both minimal n-valuedness and maximal paraconsistency, C_3 being exactly LP. And what is more, we prove that, in case $n = [>]4$,like for LP [resp., HZ/LA], there are just two proper consistent extensions of C_n — the classical one, defined by the two-valued submatrix $\mathcal{A}_{n:2}$ of the n-valued matrix $\mathcal{A}_n$ defining C_n and relatively axiomatized by the *Resolution/"Modus Ponens"* rule /"for *material* implication" [or ({un}like HZ/LA {resp., LP}) by a single axiom], and its proper sublogic, defined by the direct product of $\mathcal{A}_n$ and $\mathcal{A}_{n:2}$ (in which case having the same theorems as C_n has, and so not being an axiomatic extension of C_n) and relatively axiomatized by the *Ex Contradictione Quodlibet* rule. Finally, we find both a sequent axiomatization of C_n with Cut Elimination Property that is algebraizable iff $n \neq 4$, C_n as such being algebraizable iff $n > 4$, in which case it is equivalent to its sequent axiomatization, and a finite Hilbert-style one as well as, in case $n > 4$, finite equational axiomatizations of the discriminator variety equivalent to both C_n and its sequent axiomatization.

Key words: logic; matrix; extension; sequent; calculus; discriminator.
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§1. Introduction. Appearance of any logic/calculus satisfying a property P inevitably raises the question whether it can not be enhanced (by extending with new — viz., non-derivable — rules [without premises]) but with retaining the property P, in which case it is said to be *[axiomatically] maximal* P.

Within the framework of Paraconsistent Logic, P is paraconsistency — viz., refuting the *Ex Contradictione Quodlibet* rule. Then, maximal paraconsistency (versus it axiomatic version first observed in [24] for P^1) was first discovered in [12] for the *logic of paradox LP* [10] and then also proved in [16] for HZ [4] and for arbitrary expansions of the *logic of antinomies LA* [1] in [19]. And what is more, it has been proved for arbitrary conjunctive paraconsistent subclassical three-valued logics (including all the *particular* logics mentioned above, and so providing a first proof of the maximal paraconsistency of P^1; cf. the reference [Pyn95 b] of [12]).

On the other hand, according to [20], there are *minimally* four-valued maximally paraconsistent (even subclassical) logics. This, first, has definitely shown that the maximal paraconsistency is not at all a prerogative of three-valued logics, and, second, has inevitably raised the question whether there is any limit $n > 0$ such that any *minimally* m-valued paraconsistent logic is maximally paraconsistent, for no $m > n$. The stipulation "minimally" is essential here, simply because any n-valued matrix has an m-valued strict surjective homomorphic counter-image, for any $m > n$, in which case any n-valued logic is equally m-valued, and so the above three-valued maximally paraconsistent instances would *immediately* yield the negative answer to the question under consideration but without the stipulation involved. The primary purpose of this paper is to give a (negative) answer with taking the mentioned stipulation into account, the secondary one being to find the lattices of extensions, both sequent and finite Hilbert-style axiomatizations as well as finite equational axiomatizations of equivalent varieties (if any) of proposed instances, for these, being closely related to LP, have appeared quite interesting.

The rest of the paper is as follows. The exposition of the material is perfectly self-contained (of course, modulo very basic issues of Set and Lattice Theories, Universal Algebra and Mathematical Logic to be consulted in standard mathematical handbooks like [2, 6, 7] or fundamental papers like [5]). We entirely follow the standard conventions (most of which have become a part of logical and algebraic folklore constituting foundations of General Logic) adopted in [20], to Sections 2 and 3 of which the reader is referred just in case it is necessary. Section 2 is then to provide certain key issues proving beyond the scopes of the mentioned study, those appearing therein being still briefly recalled for the sake of self-containity. Finally, Section 3 is devoted to the main results of the paper.

§2. Preliminaries. As usual (cf., e.g., [7]), natural numbers (including 0) are treated as ordinals (viz., sets of lesser natural numbers), the set of all them being denoted by ω, while functions are viewed as binary relations with the left/right components of their elements as their arguments/values, respectively, but with standard (viz., left-|right-hand) writing functions|arguments, respectively, in which case though $(f \circ g)(a) = g(f(a))$, where f and g are functions with $(\mathrm{img}\, f) \subseteq (\mathrm{dom}\, g)$ and $a \in (\mathrm{dom}\, f) = \mathrm{dom}(f \circ g)$, whereas singletons are identified with their unique elements, unless any confusion is possible. Likewise, given any set S (and any equivalence relation θ on it), let $\wp_{[\alpha]}(S)$ [where $\alpha \subseteq \omega$] be the set of all subsets of S [of cardinality in α] (as well as both $\nu_\theta \triangleq \{\langle s, \theta[\{s\}]\rangle \mid s \in S\}$

and $(T/\theta) \triangleq \nu_\theta[T]$, where $T \subseteq S$). Then, given any $f : S \to S$, set $f^1 \triangleq f$ and $f^0 \triangleq \Delta_S \triangleq \{\langle s, s\rangle \mid s \in S\}$, binary relations of the latter kind being referred to as *diagonal*. Next, any S-tuple (viz., a function with domain S) is normally written in the sequence form $\bar{t}$, its s-th component (viz., the value on argument $s \in S$) being written as t_s. Further, set $S^+ \triangleq \bigcup_{i\in(\omega\setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples/"comma separated sequences". Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^+$, put:

$$(\diamond\bar{a}) \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}\restriction(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Finally, an *enumeration of* S is any bijection from its cardinality $|S|$ onto S.

In general, to unify algebraic notations, unless otherwise specified, algebra[ic system]s [cf. [6]; (including logical matrices; cf. [5])] are denoted by capital Fraktur [resp. Calligraphic] letters, their underlying sets (viz., carriers) [resp., underlying algebras (viz., algebra reducts)] being denoted by corresponding capital Italic [resp., Fraktur] letters.

Let Σ be a *(propositional/sentential) language|signature* constituted by *(propositional/sentential) connectives* to be viewed as function symbols. Then, the absolutely-free Σ-algebra, freely generated by the set $V_\omega \triangleq \{x_i \mid i \in \omega\}$ of *(propositional/sentential) variables*, is denoted by $\mathfrak{Fm}_\Sigma$, the standard algebra superscript being normally omitted in writing its operations, elements of its carrier Fm_Σ being called *(propositional/sentential) Σ-formulas* to be viewed as Σ-terms. As usual, any couple $\langle\phi, \psi\rangle$ of Σ-formulas is viewed as a Σ*-equation/-identity* to be written in the standard equational form $\phi \approx \psi$. Likewise, a *(two-side)* Σ*-sequent* is any couple $\langle\bar{\phi}, \bar{\psi}\rangle$ of finite tuples of Σ-formulas normally written in the standard sequential form $\bar{\phi} \vdash \bar{\psi}$.

Recall that a *(ternary) discriminator for* a Σ-algebra[1] $\mathfrak{A}$ is any Σ-formula δ with at most three variables x_0, x_1 and x_2 such that

$$\delta^{\mathfrak{A}}[x_i/a_i]_{i\in 3} = \begin{cases} a_2 & \text{if } a_0 = a_1, \\ a_0 & \text{otherwise,} \end{cases}$$

for all $\bar{a} \in A^3$, in which case, for any $\theta \in (\mathrm{Con}(\mathfrak{A}) \setminus \{\Delta_A\})$, any $\langle a, b\rangle \in (\theta \setminus \Delta_A)$ and any $c \in A$, we have $c = \delta^{\mathfrak{A}}(a, b, c)\ \theta\ \delta^{\mathfrak{A}}(a, a, c) = a$, and so we get $\theta = A^2$ (in particular, $\mathfrak{A}$ has no non-diagonal congruence other than A^2).

[1]Algebras with discriminator are also referred to as *quasi-primal* (cf., e.g., [9]).

As usual, *[bounded] distributive lattices* (cf., e.g., [2]) are supposed to be of the signature $\Sigma_{+[01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}])$ with binary $\wedge$ (conjunction/meet) and $\vee$ (disjunction/join) [as well as nullary $\bot$ (falsehood/zero) and $\top$ (truth/unit)]. Given any $n > 0$, by $\mathfrak{D}_{n[01]}$ we denote the [bounded] distributive lattice given by the chain poset $\langle n, \leqslant\rangle$. In general, given any signature $\Sigma \supseteq \Sigma_+$ and any $\phi, \psi \in \mathrm{Fm}_\Sigma$, either $\phi \lessapprox \psi$ or $\psi \gtrapprox \phi$ stands for the Σ-equation $(\phi \wedge \psi) \approx \phi$. Likewise, given any Σ-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma_+$ is a lattice, "the partial ordering"/"[prime] ideals|filters" of the latter "is denoted by $\leqslant^{\mathfrak{A}}$"/"are called those of $\mathfrak{A}$", respectively. Let $\Sigma_{\sim[01]} \triangleq (\Sigma_{+[01]} \cup \{\sim\})$ be the signature with unary $\sim$ (negation). Then, a *[bounded] Kleene lattice* is any $\Sigma_{\sim[01]}$-algebra $\mathfrak{A}$, the $\Sigma_{+[01]}$-reduct of which is a [bounded] distributive lattice and which satisfies the identities:

$$\sim\sim x_0 \approx x_0, \tag{2.1}$$

$$\sim(x_0 \vee x_1) \approx \sim x_0 \wedge \sim x_1, \tag{2.2}$$

$$\sim(x_0 \wedge x_1) \approx \sim x_0 \vee \sim x_1, \tag{2.3}$$

$$(x_0 \wedge \sim x_0) \lessapprox (x_1 \vee \sim x_1) \tag{2.4}$$

[in which case it satisfies the identities:

$$\sim\bot \approx \top, \tag{2.5}$$

$$\sim\top \approx \bot, \tag{2.6}$$

and also called a *Kleene/Boolean algebra* (following a traditional terminology; cf., e.g., [2]) /"whenever each element of it is Boolean", any $a \in A$ being referred to as *Boolean*, whenever $(a \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} a) = \bot^{\mathfrak{A}}$, that is (in view of (2.1), (2.2), (2.3), (2.5) and (2.6)), $(a \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} a) = \top^{\mathfrak{A}}$, the set $\flat^{\mathfrak{A}}$ of all Boolean elements of $\mathfrak{A}$ forming a Boolean subalgebra of it]. Given any $n > 0$, by $\mathfrak{K}_{n[01]}$ we denote the [bounded] Kleene lattice such that $(\mathfrak{K}_{n[01]}{\restriction}\Sigma_{+[01]}) \triangleq \mathfrak{D}_{n[01]}$ and, for all $i \in n$, $\sim^{\mathfrak{K}_{n[01]}} i \triangleq (n - 1 - i)$.

Next, elements/subsets of $\wp_{[1]}(\mathrm{Fm}_\Sigma) \times \mathrm{Fm}_\Sigma$ are called $\{$*Hilbert-style*$\}$ *(propositional|sentential) [axiomatic] Σ-rules/-calculi*, respectively, any [axiomatic] Σ-rule $\langle\Gamma, \varphi\rangle$ being normally written in the standard inline $\Gamma \to \varphi$ or displayed $\frac{\Gamma}{\varphi}$ forms and semantically viewed as the infinitary basic Horn formula $(\bigwedge \Gamma) \to \varphi$ of the first-order signature $\Sigma \cup \{D\}$ with single unary *truth* predicate D — under the identification of any Σ-formula ψ with the atomic first-order formula $D(\psi)$ — [as well as being referred to as a *(propositional|sentential) Σ-axiom* and identified with φ]. Then, given any $\phi, \psi \in \mathrm{Fm}_\Sigma$, set

$$\frac{\phi}{\psi}\updownarrow \triangleq \{\frac{\phi}{\psi}, \frac{\psi}{\phi}\}.$$

Likewise, as usual, any Σ-sequent $\bar{\phi} \vdash \bar{\psi}$ is semantically treated as the first-order disjunct $\bigvee(\neg[D[\mathrm{img}\,\bar{\phi}] \cup D[\mathrm{img}\,\bar{\psi}])$ of the first-order signature involved.

Unless otherwise specified, throughout the paper, $\wr/\diamond$ is supposed to be a (possibly, secondary) unary/binary connective of Σ (i.e., a Σ-formula with at most one/two variable/s x_0/" and x_1").

Now, recall that a *(propositional/sentential) Σ-logic* (cf., e.g., [5]) is any closure operator C over Fm_Σ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}_\Sigma$ and all $\sigma \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{Fm}_\Sigma)$. Then, a Σ-rule $\Gamma \to \varphi$ is said to be *satisfied in* C, provided $\varphi \in C(\Gamma)$, Σ-axioms satisfied in C being called its *theorems*. Next, a Σ-logic C' is said to be a *[proper] extension of* C ($C \subseteq [\subsetneq]C'$, in symbols), provided $[C' \neq C$ and$]$ $C(X) \subseteq C'(X)$, for all $X \subseteq \mathrm{Fm}_\Sigma$, in which case C is referred to as a *[proper] sublogic of* C'. Then, C' is said to be *axiomatized by* a[n axiomatic] Σ-calculus $\mathcal{C}$ *(relatively to C)*, provided C' is the least {under the extension partial ordering $\subseteq$} Σ-logic (being an extension of C and) satisfying every Σ-rule in $\mathcal{C}$ [(in which case C' is called an *axiomatic extension of* C)]. Further, C is said to be *$\diamond$-conjunctive*|*-disjunctive*, provided $C(X \cup \{\phi \diamond \psi\}) = C(C(X \cup \{\phi\})(\cup|\cap)C(X \cup \{\psi\}))$, for all $X \subseteq \mathrm{Fm}_\Sigma$ and all $\phi, \psi \in \mathrm{Fm}_\Sigma$. Likewise, C is said to be *weakly $\diamond$-implicative*, provided it satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \diamond x_1\} \to x_1 \tag{2.7}$$

and *has Deduction* (viz,. *Herbrand*; cf. [7]) *theorem (DT/HT) with respect to* $\diamond$ in the sense that, for all $\phi \in X \subseteq \mathrm{Fm}_\Sigma$ and all $\psi \in C(X)$, it holds that $(\phi \diamond \psi) \in C(X \setminus \{\phi\})$, in which case the following axioms:

$$(x_0 \diamond x_0), \tag{2.8}$$

$$(x_0 \diamond (x_1 \diamond x_0), \tag{2.9}$$

$$(x_0 \diamond x_1) \diamond ((x_1 \diamond x_2) \diamond (x_0 \diamond x_2)) \tag{2.10}$$

are satisfied in C. Then, C is said to be *(strongly) $\diamond$-implicative*, whenever it is weakly so and satisfies the *Peirce Law* axiom (cf. [8]):

$$(((x_0 \sqsupset x_1) \sqsupset x_0) \sqsupset x_0). \tag{2.11}$$

Furthermore, C is said to be *(maximally) [$\wr$-para]consistent*, provided $x_1 \notin C(\varnothing[\cup\{x_0, \wr x_0\}])$ (and C has no proper [$\wr$-para]consistent extension). Finally, given any $\Sigma' \subseteq \Sigma$, we have the Σ'-logic C', given by $C'(X) \triangleq (C(X) \cap \mathrm{Fm}_{\Sigma'})$, for all $X \subseteq \mathrm{Fm}_{\Sigma'}$, called the *$\Sigma'$-fragment of* C, in which case C is referred to as a *(Σ-)expansion of* C'.

As usual, any *(logical) Σ-matrix* $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$ with its *underlying* Σ-algebra $\mathfrak{A}$ and its *truth predicate* (viz., the set of its *distinguished* values) $D^{\mathcal{A}} \subseteq A$ (cf., e.g., [5], to which the reader is referred for the conception of the *logic* $\mathrm{Cn}_{\mathcal{A}}$ *of/"defined by"* $\mathcal{A}$) is treated as a first-order model structure (viz, an algebraic system; cf. [6], to which the reader is referred for notions of [sub]direct product|power, subsystem, etc.) of the first-order signature $\Sigma \cup \{D\}$, in which case any Σ-rule is true (viz., satisfied) in $\mathcal{A}$ iff it is satisfied in $\mathrm{Cn}_{\mathcal{A}}$. Then, $\mathcal{A}$ is said to be *$\diamond$-conjunctive/-disjunctive*, provided, for all $a, b \in A$, $(a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff both/either $a \in D^{\mathcal{A}}$ and/or $b \in D^{\mathcal{A}}$, "that is,"/"in which case" its logic is so, respectively. Likewise, $\mathcal{A}$ is said to be *$\diamond$-implicative*, provided, for all $a, b \in A$, $(a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff either $a \notin D^{\mathcal{A}}$ or $b \in D^{\mathcal{A}}$, in which case it is $\vee_\diamond$-disjunctive, where $(x_0 \vee_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$, and so its logic is strongly $\diamond$-implicative, for $(2.11) = ((x_0 \diamond x_1) \vee_\diamond x_0)$. Next, $\mathcal{A}$ is said to be *[$\wr$-para]consistent*, provided $A \neq D^{\mathcal{A}}$ [and $\{a, \wr^{\mathfrak{A}} a\} \subseteq D^{\mathcal{A}}$, for some $a \in A$], that is, the logic of it is so. Likewise, $\mathcal{A}$ is said to be *$\wr$-negative*, provided $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, for all $a \in A$. Further, according to [17], an *equality determinant for $\mathcal{A}$* is any set $\Im$ of Σ-formulas with at most one variable x_0 such that any $a, b \in A$ are equal, whenever, for every $\iota \in \Im$, it holds that $(\iota^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\iota^{\mathfrak{A}}(b) \in D^{\mathcal{A}})$. In that case, given any $a \in A$, set $\Im_{\mathcal{A},a,+/-} \triangleq \{\iota \in \Im \mid \iota^{\mathfrak{A}}(a) \in / \notin D^{\mathcal{A}_n}\}$, respectively. Furthermore, according to [15], a set ε of Σ-equations with at most one variable x_0 is said to *define (equationally) truth [predicate] of/in $\mathcal{A}$*, provided, for all $a \in A$, $a \in D^{\mathcal{A}}$ iff $\mathfrak{A} \models (\bigwedge \varepsilon)[x_0/a]$. Likewise, according to Appendix A of [19], a set ε of Σ-equations with at most two variable x_0 and x_1 is called an *equational implication for $\mathcal{A}$*, provided, for all $a, b \in A$, $(a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})$ iff $\mathfrak{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/b]$. Next, a *congruence of* $\mathcal{A}$ is any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$ (in which case we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$), the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$, $\mathcal{A}$ being said to be *simple*, whenever $\mathrm{Con}(\mathcal{A}) = \{\Delta_A\}$. Then, the transitive closure $\eth(\mathcal{A})$ of $\bigcup \mathrm{Con}(\mathcal{A})$ is the greatest congruence of $\mathcal{A}$. Further, $\mathcal{A}$ is said to be a *model of* a Σ-logic C, provided its logic is an extension of C, the class of all them being denoted by $\mathrm{Mod}(C)$. Furthermore, $\mathcal{A}$ is said to be *finite[ly-generated]/"generated by $B \subseteq A$"|n-valued*, where $n > 0$, whenever $\mathfrak{A}$ is so|n-element, respectively, the logics of n-valued Σ-matrices being well-known to be finitary (cf. [5]) and referred to as *[minimally] (uniform) n-valued* [unless they are m-valued, for any $0 < m < n$]. Then, both two-valued and $\wr$-negative Σ-matrices are said to be *$\wr$-classical*, [sublogics of] their logics being referred to as *$\wr$-[sub]classical*. In addition, $\mathcal{A}$ is said to be *false-singular*, provided $A \setminus D^{\mathcal{A}}$ has no more than one element. Finally, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* $(\mathcal{A} \restriction \Sigma') \triangleq \langle \mathfrak{A} \restriction \Sigma', D^{\mathcal{A}} \rangle$, then defining the Σ'-fragment of the logic of $\mathcal{A}$.

Given Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ such that the set $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B}) \triangleq \{h \in \hom(\mathfrak{A}, \mathfrak{B}) \mid [h[A] = B,] D^{\mathcal{A}} \subseteq h^{-1}[D^{\mathcal{B}}] (\supseteq D^{\mathcal{A}})\}$ of all *(strict) [surjective] homomorphisms from $\mathcal{A}$ [on]to $\mathcal{B}$* is not empty (in which case $\mathcal{A}$ is $\diamond$-conjunctive|-disjunctive|-implicative if[f] $\mathcal{B}$ is so, while the logic of $\mathcal{A}$ is a [non-proper] extension of the one of $\mathcal{B}$; cf. (2.2) of [20], whereas $(\ker h) \in \mathrm{Con}(\mathcal{A})$, and so h is injective, whenever $\mathcal{A}$ is simple; cf. Remark 2.2 and Corollary 2.3 of [20]), injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ being called *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, $\mathcal{B}|\mathcal{A}$ is referred to as a *(strict) [surjective] homomorphic image|counter-image of* $\mathcal{A}|\mathcal{B}$, respectively. Then, the class of all "(consistent) submatrices"/"strict surjective homomorphic [counter-]images" of members of any class M of Σ-matrices is denoted by $(\mathbf{S}_{(*)}/\mathbf{H}^{[-1]})(\mathsf{M})$, respectively.

2.1. False-singular matrices.

2.1.1. *Conjunctive matrices.*

Lemma 2.1. *Let $\mathcal{A}$ be a false-singular $\diamond$-conjunctive Σ-matrix, $f \in (A \setminus D^{\mathcal{A}})$, I a finite set, $\overline{\mathcal{B}}$ an I-tuple constituted by consistent submatrices of $\mathcal{A}$ and $\mathcal{D}$ a subdirect product of it. Then, $(I \times \{f\}) \in D$.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in D$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in D \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{B}_j$ is a consistent submatrix of the false-singular Σ-matrix $\mathcal{A}$, we have $f \in B_j = \pi_j[D]$. Hence, there is some $b \in D$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $a \in D$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both $\diamond$-conjunctive and false-singular, we have $D \ni c \triangleq (a \diamond^{\mathfrak{D}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $D \ni (I \times \{f\})$, as required. $\dashv$

2.1.2. *Implicative matrices.*

Lemma 2.2. *Let $\mathcal{A}$ be a false-singular Σ-matrix and C the logic of it. Then, the following are equivalent:*

(i) *C is stronly $\diamond$-implicative;*
(ii) *C is weakly $\diamond$-implicative;*
(iii) *C (viz., $\mathcal{A}$) satisfies* (2.8), (2.9) *and* (2.7)*;*
(iv) *$\mathcal{A}$ is $\diamond$-implicative.*

Proof. First, (iv/ii)$\Rightarrow$(i/iii) are immediate. Next, (ii) is a particular case of (i). Finally, assume (iii) holds. Consider any $a, b \in A$. Then, by (2.7) and (2.9), $(a \diamond^{\mathfrak{A}} b) \in / \notin D^{\mathcal{A}}$, whenever $b \in / \notin D^{\mathcal{A}} / \ni a$. Now, assume $a \notin D^{\mathcal{A}} \not\ni b$, in which case $a = b$, and so, by (2.8), $D^{\mathcal{A}} \ni (a \diamond^{\mathfrak{A}} a) = (a \diamond^{\mathfrak{A}} b)$. Thus, (iv) holds. $\dashv$

§3. Main results. Fix any $n > 2$.

Let $N_{n[-]} \triangleq \{i \in ((n-1) \setminus 1) \mid (2 \cdot i) < (n[-1])\}$, $\Sigma_n \triangleq (\Sigma_{\sim[01]} \cup \{\partial_i \mid i \in N_{n-}\} \cup \{\nabla_j \mid n > 4, j \in N_n\})$ [whenever $n > 3$] the signature with unary connectives in $\Sigma_n \setminus \Sigma_{\sim[01]}$, $\mathcal{A}_n$ the Σ_n-matrix with $(\mathfrak{A}_n {\restriction} \Sigma_{\sim[01]}) \triangleq \mathfrak{K}_{n[01]}$,

$D^{\mathcal{A}_n} \triangleq (n \setminus 1)$ and, for all $k \in n$:

$$\partial_i^{\mathfrak{A}_n} k \triangleq \begin{cases} n-1 & \text{if } i < k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $i \in N_{n-}$, as well as, in case $n > 4$:

$$\nabla_j^{\mathfrak{A}_n} k \triangleq \begin{cases} j & \text{if } k \in ((n-1) \setminus 1), \\ k & \text{otherwise,} \end{cases}$$

for all $j \in N_n$, and C_n the logic of $\mathcal{A}_n$. Then, $\mathcal{A}_n$ is false-singular (as well as both $\wedge$-conjunctive, $\vee$-disjunctive and $\sim$-paraconsistent, and so is C_n). Moreover, C_3 is the *logic of paradox LP* [10] (cf. [12]). And what is more, $\{0, n-1\}$ forms a subalgera of $\mathfrak{A}_n$, in which case $\mathcal{A}_{n:2} \triangleq (\mathcal{A}_n \restriction \{0, n-1\})$ is a $\sim$-classical model of C_n, and so this is $\sim$-subclassical (more precisely, it is a sublogic of the $\sim$-classical logic C_n^{PC} of $\mathcal{A}_{n:2}$), while $h_{n:2} \triangleq (((n \setminus 1) \times \{n-1\}) \cup \{\langle 0, 0\rangle\}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}_n \restriction \Sigma_+, \mathcal{A}_{n:2} \restriction \Sigma_+)$, so the Σ_+-fragment of C_n is equal to that of C_n^{PC}, whereas $h_{n:3} \triangleq ((((n-1) \setminus 1) \times \{1\}) \cup \{\langle 0, 0\rangle, \langle n-1, 2\rangle\}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}_n \restriction \Sigma_\sim, \mathcal{A}_3)$, in which case $LP = C_3$ is the $\Sigma_\sim$-fragment of C_n, and so, being then defined by $\mathcal{A}_n \restriction \Sigma_\sim$, is n-valued.

Lemma 3.1. *$\Im_n \triangleq (\{x_0, \sim x_0\} \cup \{\partial_i \sim^j x_0 \mid i \in N_{n-}, j \in 2\})$ is an equality determinant for $\mathcal{A}_n$.*

Proof. Consider any $k, l \in n$ such that $k < l$ and the following complementary cases:

- $0 \in \{k, l\}$,
  in which case $k = 0 \neq l$, and so $k \notin D^{\mathcal{A}_n} \ni l$.
- $0 \notin \{k, l\}$.
  Consider the following complementary subcases:
  - $(n-1) \in \{k, l\}$,
    in which case $(n-1-k) \neq 0 = (n-1-l)$, and so $\sim^{\mathfrak{A}_n} k = (n-1-k) \in D^{\mathcal{A}_n} \not\ni (n-1-l) = \sim^{\mathfrak{A}_n} l$.
  - $(n-1) \notin \{k, l\}$.
    Consider the following complementary subsubcases:
    * $l \in N_n$,
      in which case $l > k \in N_{n-}$, and so $\partial_k^{\mathfrak{A}_n} k = 0 \notin D^{\mathcal{A}_n} \ni (n-1) = \partial_k^{\mathfrak{A}_n} l$.
    * $l \notin N_n$,
      in which case $(n-1-k) > (n-1-l) \in N_{n-}$, and so $\partial_{n-1-l}^{\mathfrak{A}_n} \sim^{\mathfrak{A}_n} k = \partial_{n-1-l}^{\mathfrak{A}_n}(n-1-k) = (n-1) \in D^{\mathcal{A}_n} \not\ni 0 = \partial_{n-1-l}^{\mathfrak{A}_n}(n-1-l) = \partial_{n-1-l}^{\mathfrak{A}_n} \sim^{\mathfrak{A}_n} l$. ⊣

To unify further notations, set $C_n^{-\mathrm{PC}} \triangleq C_n$ and $\mathcal{A}_{n-:2} \triangleq \mathcal{A}_n$.

Now, we are in a position to prove the following key lemma "killing two birds — the minimal n-valuedness and maximal $\sim$-paraconsistency of C_n — with one stone":

LEMMA 3.2 (Key Lemma). *Let $\mathcal{B}$ be a [$\sim$-para]consistent model of C_n. Then, $\mathcal{A}_{n[-]:2}$ is a strict surjective homomorphic image of a submatrix of $\mathcal{B}$.*

PROOF. Then, there is [resp., are some $a \in D^{\mathcal{B}}$ and] some $b \in (B \setminus D^{\mathcal{B}})$ [such that $\sim^{\mathfrak{B}} a \in D^{\mathcal{B}}$], in which case the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{[a,]b\}$ is a finitely-generated [$\sim$-para]consistent model of C_n, and so, by Lemma 2.7 of [20], there are some finite set I, some $\overline{\mathcal{E}} \in \mathbf{S}_*(\mathcal{A}_n)^I$, some subdirect product $\mathcal{F}$ of it, some Σ_n-matrix $\mathcal{G}$ and some $(g|h) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}|\mathcal{F}, \mathcal{G})$. In that case, $\mathcal{F}$ is [$\sim$-para]consistent, and so $I \neq \varnothing$. Then, by Lemma 2.1, $e \triangleq (I \times \{0\}) \in F$, in which case $F \ni \sim^{\mathfrak{F}} e = (I \times \{n-1\})$. [Moreover, there is some $c \in D^{\mathcal{F}}$ such that $\sim^{\mathfrak{F}} c \in D^{\mathcal{F}}$, in which case $c \in ((n-1) \setminus 1)^I$, and so $d \triangleq (c \wedge^{\mathfrak{F}} c) \in N_n^I$. Consider any $i \in N_n$ and the following complementary cases:

- $n \leqslant 4$,
  in which case $N_n = \{1\}$, and so $i = 1$. Then, $F \ni d = (I \times \{i\})$.
- $n > 4$,
  in which case $\nabla_i \in \Sigma_n$, and so $F \ni \nabla_i^{\mathfrak{F}} d = (I \times \{i\})$.

Thus, in any case, $F \ni (I \times \{i\})$. On the other hand, for every $j \in (((n-1) \setminus 1) \setminus N_n)$, $(n-1-j) \in N_n$, in which case $F \ni (I \times \{n-1-j\})$, and so $F \ni \sim^{\mathfrak{F}}(I \times \{n-1-j\}) = (I \times \{j\})$.] In this way, $\{I \times \{k\} \mid k \in A_{n[-]:2}\} \subseteq F$. Hence, as $I \neq \varnothing$, $f \triangleq \{\langle k, I \times \{k\} \rangle \mid k \in A_{n[-]:2}\}$ is an embedding of $\mathcal{A}_{n[-]:2}$ into $\mathcal{F}$, in which case, by Lemmas 3.2, 3.3 of [20] and 3.1, $f \circ h$ is that into $\mathcal{G}$, and so $\mathrm{img}(f \circ h)$ forms a subalgebra of $\mathfrak{G}$. Then, $H \triangleq g^{-1}[\mathrm{img}(f \circ h)]$ forms a subalgebra of $\mathfrak{D}$, while $\mathcal{H} \triangleq (\mathcal{D} \restriction H)$ is a submatrix of $\mathcal{B}$, whereas $((g \restriction H) \circ (h \circ f)^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{A}_{n[-]:2})$. ⊣

THEOREM 3.3 (cf. Theorem 2.1 of [12] for $n = 3$). *$C_n^{[-]\mathrm{PC}}$ is maximally [$\sim$-para]consistent.*

PROOF. Let C be a [$\sim$-para]consistent extension of $C_n^{[-]\mathrm{PC}}$, in which case $x_1 \notin T \triangleq C(\varnothing[\cup\{x_0, \sim x_0\}])$, and so, by the structurality of C, $\mathcal{B} \triangleq \langle \mathfrak{Fm}_\Sigma, T \rangle$ is a [$\sim$-para]consistent model of C (in particular, of C_n). Hence, by Lemma 3.2, $\mathcal{A}_{n[-]:2} \in \mathrm{Mod}(C)$, in which case $C_n^{[-]\mathrm{PC}}$ is an extension of C, and so $C = C_n^{[-]\mathrm{PC}}$, as required. ⊣

THEOREM 3.4. *C_n is minimally n-valued.*

PROOF. Let $\mathcal{B}$ be a Σ_n-matrix defining C_n, in which case, as C_n is $\sim$-paraconsistent, $\mathcal{B} \in \mathrm{Mod}(C_n)$ is $\sim$-paraconsistent, and so, by Lemma 3.2, there are some submatrix $\mathcal{D}$ of $\mathcal{B}$, being a strict surjective homomorphic counter-image of $\mathcal{A}_n$. In this way, $n = |A_n| \leqslant |D| \leqslant |B|$, as required. $\dashv$

Further, we argue that each of the supplementary unary operators in $\Sigma_n \setminus \Sigma_\sim$ is necessary for both Theorems 3.3 and 3.4 to hold.

Proposition 3.5. *Let $i \in N_{n-}$, $\Sigma' \subseteq (\Sigma_n \setminus \{\partial_i\})$, $\mathcal{A}'_n \triangleq (\mathcal{A}_n \restriction \Sigma')$ and C'_n the logic of $\mathcal{A}'_n$. Then, C'_n is non-minimally n-valued.*

Proof. Then, $(i+1) < (n-1)$, for, otherwise, we would have $(i+1) \geqslant (n-1) > (2 \cdot i)$, in which case we would get $i < 1$, and so would eventually get $i = 0 \notin N_{n-}$. Thus, $D \triangleq \{i, i+1, n-1-i, n-2-i\} \subseteq ((n-1) \setminus 1) \subseteq D^{\mathcal{A}_n}$. In particular, in case $n > 4$, $\nabla_j^{\mathfrak{A}_n} d = j$, for all $d \in D$ and all $j \in N_n$. Let $\vartheta \triangleq (\Delta_n \cup \{\langle i, i+1\rangle, \langle n-1-i, n-2-i\rangle\})$. Consider any $\langle a, b\rangle \in (\vartheta \setminus \Delta_n)$, in which case $\min(a, b) = (\max(a, b) - 1)$, and so, for all $c \in n$, we have $(a(\wedge|\vee)^{\mathfrak{A}_n} c) = (b(\wedge|\vee)^{\mathfrak{A}_n} c) = (c(\wedge|\vee)^{\mathfrak{A}_n} a) = (c(\wedge|\vee)^{\mathfrak{A}_n} b) = c$, if $c \leqslant | \geqslant (\min|\max)(a, b)$, and both $(a(\wedge|\vee)^{\mathfrak{A}_n} c) = a = (c(\wedge|\vee)^{\mathfrak{A}_n} a)$ and $(b(\wedge|\vee)^{\mathfrak{A}_n} c) = b = (c(\wedge|\vee)^{\mathfrak{A}_n} b)$, otherwise. Then, in case $n > 4$, for all $j \in N_n$, $\langle \nabla_j^{\mathfrak{A}_n} a, \nabla_j^{\mathfrak{A}_n} b\rangle \in \Delta_n \subseteq \vartheta$. Now, consider any $k \in (N_{n-} \setminus \{i\})$. Let us prove, by contradiction, that $k \neq (n-2-i)$. For suppose $k = (n-2-i)$, in which case, as $k \in N_{n-}$, we have $(2 \cdot (n-2-i)) \leqslant (n-2)$, and so we get $(n-2) \leqslant (2 \cdot i)$. Conversely, as $i \in N_{n-}$, we also have $(2 \cdot i) \leqslant (n-2)$, in which case we get $(n-2) = (2 \cdot i)$, and so we eventually get $k = (n-2-i) = ((2 \cdot i) - i) = i$. This contradiction shows that $k \neq (n-2-i)$. Thus, we have $k \neq \min(a, b)$. Therefore, $\partial_k^{\mathfrak{A}_n} a = 1 = \partial_k^{\mathfrak{A}_n} b$, if $k \leqslant \min(a, b)$, and $\partial_k^{\mathfrak{A}_n} a = 0 = \partial_k^{\mathfrak{A}_n} b$, otherwise. Hence, $\langle \partial_k^{\mathfrak{A}_n} a, \partial_k^{\mathfrak{A}_n} b\rangle \in \Delta_n \subseteq \vartheta$. Finally, we clearly have $\langle {\sim}^{\mathfrak{A}_n} a, {\sim}^{\mathfrak{A}_n} b\rangle \in \vartheta$. In this way, $\vartheta \supseteq \Delta_n$ is closed under unary algebraic operations of $\mathfrak{A}'_n$. Then, the transitive closure θ of $\vartheta \cup \vartheta^{-1}$ is a non-diagonal congruence of $\mathcal{A}'_n$, in which case C'_n is defined by $\mathcal{B} \triangleq (\mathcal{A}'_n/\theta)$, for $\nu_\theta \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}'_n, \mathcal{B})$, and so is not minimally n-valued, for $|B| < |A'_n| = n$, as required. $\dashv$

In general, when $N_{n|(n-)} \neq \varnothing$ (i.e., $1 \in N_{n|(n-)}$, that is, $n > (2|3)$), let $(l|m)_n \triangleq \max(N_{n|(n-)}) \in N_{n|(n-)}$, in which case $N_{n|(n-)} = (((l|m)_n + 1) \setminus 1)$, respectively.

Proposition 3.6. *Suppose $n > 4$. Let $i \in N_n$, $(\Sigma_\sim \cup \{\partial_j \mid j \in N_{n-}\}) \subseteq \Sigma'' \subseteq (\Sigma_n \setminus \{\nabla_i\})$, $\mathcal{A}''_n \triangleq (\mathcal{A}_n \restriction \Sigma'')$ and C''_n the logic of $\mathcal{A}''_n$. Then, C''_n is non-maximally $\sim$-paraconsistent.*

Proof. In that case, $B \triangleq (A_n \setminus \{i, n-1-i\}) \ni 0$ forms a subalgebra of $\mathfrak{A}''_n$, for $(n-1-i) \in (((n-1) \setminus 1) \setminus N_n)$, unless $(n-1-i) = i$, because $i \in N_n$, and, as $n > 4$, is not disjoint with $(n-1) \setminus 1$, in which case $\mathcal{B} \triangleq (\mathcal{A}''_n \restriction B)$ is a $\sim$-paraconsistent submatrix of $\mathcal{A}''_n$, and so defines a $\sim$-paraconsistent extension of C''_n. To prove that this is a proper extension, consider the following complementary cases:

- $i = 1$.
  Then, as $n > 4 > 3$, $1 \in N_{n-}$, in which case $\partial_1 \in \Sigma''$, and so we have the Σ''-rule $x_0 \to \partial_1 x_0$, which is true in $\mathcal{B}$ but is not true in $\mathcal{A}''_n$ under $[x_0/1]$.
- $(n-1-i) = i$.
  Then, n is odd, while $i = \frac{n-1}{2}$, whereas, for every $k \in n$, $k > m_n < (n-1-k)$ iff $k = i$, and so we have the Σ''-rule $\{\partial_{m_n} x_0, \partial_{m_n} {\sim} x_0\} \to x_1$, which is true in $\mathcal{B}$ but is not true in $\mathcal{A}''_n$ under $[x_0/i, x_1/0]$.
- $1 \neq i \neq (n-1-i)$.
  Then, $i \in N_{n-} \ni (i-1)$, and so we have the Σ''-rule $\{\partial_{i-1} x_0, {\sim}\partial_i x_0\} \to x_1$, which is true in $\mathcal{B}$ but is not true in $\mathcal{A}''_n$ under $[x_0/i, x_1/0]$. $\dashv$

Finally, the following immediate observation discloses more connections between C_n and LP, unless $n = 3$:

PROPOSITION 3.7. *Suppose $n > 3$. Let $\Sigma_\sim \subseteq \Sigma''' \subseteq (\Sigma_n \setminus \{\partial_i \mid i \in N_{n-}\})$, $\mathcal{A}'''_n \triangleq (\mathcal{A}_n \restriction \Sigma''')$, C'''_n the logic of $\mathcal{A}'''_n$ and $\mathcal{A}'''_3$ the Σ'''-expansion of $\mathcal{A}_3$ by diagonal operations. Then, $h_{n:3} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}'''_n, \mathcal{A}'''_3)$. In particular, C'''_n is defined by $\mathcal{A}'''_3$, in which case it is an n-valued term-wise definitionally equivalent expansion of LP.*

3.1. Extensions. In case $n = 3$, the lattice of extensions of $C_n = LP$ has been due to [15]. Here, we mainly explore the opposite case, when $n > 3$, and so $1 \in N_{n-}$ (in particular, $\partial_1 \in \Sigma_n$). On the other hand, its complementary subcases $n = 4$ and $n > 4$ are essentially different (especially, methodologically), so these are discussed separately but the following points, being common for all $n > 2$.

First, by C^{NP}_n we denote the least non-$\sim$-paraconsistent extension of C_n, that is, the proper extension of C_n relatively axiomatized by the *Ex Contradictione Quodlibet* rule:

$$\{x_0, {\sim}x_0\} \to x_1. \tag{3.2}$$

Likewise, by C^{MP}_n we denote the extension of C_n relatively axiomatized by the *Modus Ponens* rule for *material* implication:

$$\{x_0, {\sim}x_0 \vee x_1\} \to x_1. \tag{3.3}$$

Then, by the $\vee$-disjunctivity of C_n, C^{MP}_n is an extension of C^{NP}_n. Furthermore, (3.3), being true in $\mathcal{A}_{n:2}$, is not true in $\mathcal{A}_n \times \mathcal{A}_{n:2}$ under $[x_0/\langle 1, n-1\rangle, x_1/\langle 0, n-1\rangle]$, in which (3.2) is though true, and so, since $(\pi_0 \restriction (n \times \{0, n-1\})) \in \hom^{\mathrm{S}}(\mathcal{A}_n \times \mathcal{A}_{n:2}, \mathcal{A}_n)$, by (2.3) of [20], we have:

PROPOSITION 3.8. *$C^{\mathrm{NP}}_n(\varnothing) = C_n(\varnothing)$. In particular, C^{NP}_n is not an axiomatic extension of C_n.*

Next, by C_n^{R} we denote the extension of C_n relatively axiomatized by the *Resolution* rule (cf. [23] for roots of such terminology):

$$\{x_0 \vee x_1, {\sim} x_0 \vee x_1\} \to x_1. \tag{3.4}$$

Then, since $({\sim}(x_0 \vee x_1) \vee x_1) \in C_n(\{x_0 \vee x_1, {\sim} x_0 \vee x_1\})$, by the $\vee$-disjunctivity of C_n, we have:

$$C_n^{\mathrm{R}} = C_n^{\mathrm{MP}}. \tag{3.5}$$

Finally, remark that, unless $n = 3$, $\mathfrak{A}_{n:2}$ is the only proper subalgebra of $\mathfrak{A}_n$, because, for any $k \in ((n-1) \setminus 1)$, providing $n = [\neq]4$, it holds that $((n-1) \setminus 1) = \{({\sim}^{\mathfrak{A}_n})^j [\nabla_i^{\mathfrak{A}_n}](k \wedge^{\mathfrak{A}_n} {\sim}^{\mathfrak{A}_n} k) \mid j \in 2[, i \in N_n]\}$, while $(0(+(n-1))) = \partial_1^{\mathfrak{A}_n}(1(+1))$ whereas $(0\{+(n-1)\}) = {\sim}^{\mathfrak{A}_n}((n-1)\{-(n-1)\})$. Otherwise, $\mathcal{A}_{3:2}$ is though the only proper *consistent* submatrix of $\mathcal{A}_3$. And what is more, providing C_n is $\diamond$-implicative (viz., $\mathcal{A}_n$ is so; cf. Lemma 2.2), the *Ex Contradictione Quodlibet* axiom:

$$x_0 \diamond ({\sim} x_0 \diamond x_1), \tag{3.6}$$

being true in $\mathcal{A}_{n:2}$, for this is both ${\sim}$-negative and $\diamond$-implicative, is not true in $\mathcal{A}_n$ under $[x_0/1, x_1/0]$. In particular, by Corollary 2.9 of [20], we have:

PROPOSITION 3.9. *C_n has a proper consistent axiomatic extension iff C_n^{PC} is an axiomatic extension of C_n, in which case C_n^{PC} is a unique proper consistent axiomatic extension of C_n, and so $C_n^{\mathrm{PC}}(\varnothing) \neq C_n(\varnothing)$. In particular, providing C_n is $\diamond$-implicative (viz., $\mathcal{A}_n$ is so; cf. Lemma 2.2), C_n^{PC} is a unique proper consistent axiomatic extension of C_n and is relatively axiomatized by* (3.6)*, in which case $C_n^{\mathrm{PC}}(\varnothing) \neq C_n(\varnothing)$.*

3.1.1. *The four-valued case.*

LEMMA 3.10. *Let I be a finite set, $\overline{\mathcal{B}} \in \mathbf{S}_*(\mathcal{A}_4)^I$ and $\mathcal{D}$ a consistent non-${\sim}$-paraconsistent subdirect product of it. Then,* $\hom(\mathcal{D}, \mathcal{A}_{4:2}) \neq \varnothing$.

PROOF. Let us prove, by contradiction, that there is some $i \in I$ such that $\pi_i[D] \subseteq A_{4:2}$. For suppose $\pi_i[D] \subseteq A_{4:2}$, for no $i \in I$. By induction on the cardinality of any $J \subseteq I$, we prove that there is some $a \in (D \cap \{0,1\}^I)$ including $J \times \{1\}$. The case, when $J = \varnothing$, is by Lemma 2.1. Otherwise, take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, by induction hypothesis, there is some $b \in (D \cap \{0,1\})^I$ including $K \times \{1\}$. Moreover, as $4 \supseteq \pi_j[D] \nsubseteq A_{4:2}$, there is some $c \in D$ such that $\pi_j(c) \in (4 \setminus A_{4:2}) = \{1,2\}$. Then, $d \triangleq (c \wedge^{\mathfrak{D}} {\sim}^{\mathfrak{D}} c) \in D$, in which case, for all $i \in I$, $\pi_i(d) = 1$, if $\pi_i(c) \in \{1,2\}$ (in particular, $\pi_j(d) = 1$), and $\pi_i(d) = 0$, otherwise (in particular, $d \in \{0,1\}^I$), and so $a \triangleq (b \vee^{\mathfrak{D}} d) \in D$, while, for all $i \in I$, $\pi_i(a) = 1$, if $1 \in \{\pi_i(b), \pi_i(d)\}$, and $\pi_i(d) = 0$, otherwise (in particular, $a \in \{0,1\}^I$ includes $J \times \{1\}$, for $J = (K \cup \{j\})$). Thus, when $J = I$, there is some $a \in D \subseteq 4^I$ including (and so equal to) $(I \times \{1\})$, in which case $D \ni \sim^{\mathfrak{D}} a = (I \times \{2\} \in D^{\mathcal{D}} \ni a$, and so $\mathcal{D}$, being consistent, is $\sim$-paraconsistent. This contradiction shows that there is some $i \in I$ such that $B_i = \pi_i[D] \subseteq A_{4:2} \neq 4$, in which case B_i forms a proper subalgebra of $\mathfrak{A}_4$, and so $\mathcal{B}_i = \mathcal{A}_{4:2}$, while $(\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{B}_i)$, as required. ⊣

THEOREM 3.11. *C_4^{NP} is defined by $\mathcal{A}_4 \times \mathcal{A}_{4:2}$.*

PROOF. Consider any Σ_4-rule $\mathcal{R} = (\Gamma \to \varphi)$, where Γ is finite, not satisfied in C_4^{NP}, in which case $\varphi \notin T \triangleq C_4^{\mathrm{NP}}(\Gamma)$, and so, by the structurality of C_4^{NP}, $\mathcal{B} \triangleq \langle \mathfrak{Fm}_{\Sigma_4}, T \rangle \in \mathrm{Mod}(C_4^{\mathrm{NP}})$. Let V be the finite set of all variables *actually* occurring in $\mathcal{R}$. Then, the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by V is a finitely-generated model of C_4^{NP} (in particular, of C_4), in which $\mathcal{R}$ is not true under $[v/v]_{v \in V}$. Therefore, by Lemma 2.7 of [20], there are some finite set I, some $\overline{\mathcal{E}} \in \mathbf{S}_*(\mathcal{A}_4)^I$, some subdirect product $\mathcal{F} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case $\mathcal{R}$ is not true in $\mathcal{F} \in \mathrm{Mod}(C_4^{\mathrm{NP}})$, and so $\mathcal{F}$ is consistent but not $\sim$-paraconsistent. Hence, by Lemma 3.10, there is some $e \in \hom(\mathcal{F}, \mathcal{A}_{4:2}) \neq \varnothing$. Consider any $a \in (F \setminus D^{\mathcal{F}})$, in which case there is some $i \in I$ such that $E_i \ni \pi_i(a) \notin D^{\mathcal{E}_i} = (E_i \cap D^{\mathcal{A}_n})$, and so $f : F \to (4 \times A_{4:2}), b \mapsto \langle \pi_i(b), e(b) \rangle$ belongs to $J \triangleq \hom(\mathcal{F}, \mathcal{A}_4 \times \mathcal{A}_{4:2})$, while $f(a) \notin D^{\mathcal{A}_4 \times \mathcal{A}_{4:2}}$. In this way, $g : F \to (n \times A_{4:2})^J, b \mapsto \langle h(b) \rangle_{h \in J}$ is a strict homomorphism from $\mathcal{F}$ to $(\mathcal{A}_4 \times \mathcal{A}_{4:2})^J$, in which case $\mathcal{R}$ is not true in $\mathcal{A}_4 \times \mathcal{A}_{4:2}$, and so the finiteness of the latter completes the argument. ⊣

Note that

$$\partial_1^{\mathfrak{A}_4} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\langle \mathfrak{A}_4, \{2,3\} \rangle, \mathcal{A}_{4:2}), \tag{3.7}$$

in which case C_4^{PC} is equally defined by $\langle \mathfrak{A}_4, \{2,3\} \rangle$, and so, taking Proposition 3.9 and Theorem 3.11 into account, since $\{2,3\} \subseteq \{1,2,3\}$, we have the following four-valued analogue of Lemma 4.14 of [15] for the three-valued case:

LEMMA 3.12. $C_4^{\mathrm{PC}}(\varnothing) = C_4^{\mathrm{MP}}(\varnothing) = C_4^{\mathrm{NP}}(\varnothing) = C_4(\varnothing)$. *In particular, $C_4^{\mathrm{NP/PC}}$ is not an axiomatic extension of $C_4^{/[\mathrm{NP}]}$/", in which case C_4 has no proper consistent axiomatic extension, and so is not implicative (viz., $\mathcal{A}_4$ is not so; cf. Lemma 2.2)".*

COROLLARY 3.13. $C_4^{\mathrm{MP}} = C_4^{\mathrm{PC}}$.

PROOF. By induction on the cardinality of any $X \in \wp_\omega(\mathrm{Fm}_{\Sigma_4})$, we prove that $C_4^{\mathrm{PC}}(X) \subseteq C_4^{\mathrm{MP}}(X)$. The case, when $X = \varnothing$, is by Lemma 3.12. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_\omega(\mathrm{Fm}_{\Sigma_4})$, while $|Y| < |X|$. Consider any $\psi \in C_4^{\mathrm{PC}}(X)$, in which case, by the $\sim$-negativity and $\vee$-disjunctivity of $\mathcal{A}_{4:2}$ as well as induction hypothesis, we have $(\sim\phi \vee \psi) \in C_4^{\mathrm{PC}}(Y) \subseteq C_4^{\mathrm{MP}}(Y)$, and so by $(3.3)[x_0/\phi, x_1/\psi]$ and the structurality of C_4^{MP}, we get $\psi \in C_4^{\mathrm{MP}}(Y \cup \{\phi\}) = C_4^{\mathrm{MP}}(X)$, as required. Then, the finiteness of $\mathcal{A}_{4:2}$ completes the argument. ⊣

LEMMA 3.14. *Let C be an extension of C_4. Suppose* (3.4) *is not satisfied in C. Then, $C \subseteq C_4^{\mathrm{NP}}$.*

PROOF. Then, $x_1 \notin T \triangleq C(\{x_0 \vee x_1, {\sim}x_0 \vee x_1\})$, in which case, by the structurality of C, $\mathcal{B} \triangleq \langle \mathfrak{Fm}_{\Sigma_4}, T\rangle$ is a model of C (in particular, of C_4), and so is its finitely-generated submatrix $\mathcal{D}$ generated by $\{x_0, x_1\}$, in which (3.4) is not true under $[x_i/x_i]_{i\in 2}$. Hence, by Lemma 2.7 of [20], there are some set I and some $\mathcal{E} \in (\mathbf{H}^{-1}(\mathbf{H}(\mathcal{D})) \cap \mathbf{S}(\mathcal{A}_4^I))$, in which case (3.4) is not true in $\mathcal{E} \in \mathrm{Mod}(C) \subseteq \mathrm{Mod}(C_4)$, and so, since $\mathcal{E}$ is $\wedge$-conjunctive, for C_4 is so, while $\mathfrak{E}{\restriction}\Sigma_+$ is a distributive lattice, for $\mathfrak{A}_4{\restriction}\Sigma_+$ is so, there are some $a, b \in E$ such that $((a \wedge^{\mathfrak{E}} {\sim}^{\mathfrak{E}} a) \vee^{\mathfrak{E}} b) \in D^{\mathcal{E}} \not\ni b$. Then, $c \triangleq (a \wedge^{\mathfrak{E}} {\sim}^{\mathfrak{E}} a) \in \{0,1\}^I$. Given any $j \in 4$, set $J_j \triangleq \{i \in I \mid \pi_i(b) = j\}$. Then, $\varnothing \neq J_0 \subseteq K \triangleq \{i \in I \mid \pi_i(c) = 1\}$. Given any $\bar{k} \in 4^7$, put $(k_0 : k_1 : k_2 : k_3 : k_4 : k_5 : k_6) \triangleq (((J_3 \cap K) \times \{k_0\}) \cup ((J_3 \setminus K) \times \{k_1\}) \cup ((J_2 \cap K) \times \{k_2\}) \cup ((J_2 \setminus K) \times \{k_3\}) \cup ((J_1 \cap K) \times \{k_4\}) \cup ((J_1 \setminus K) \times \{k_5\}) \cup (J_0 \times \{k_6\})) \in 4^I$, in which case $(3 : 3 : 2 : 2 : 1 : 1 : 0) = b \in E \ni c = (1 : 0 : 1 : 0 : 1 : 0 : 1)$, and so

$$E \ni d \triangleq (\partial_1^{\mathfrak{E}} b \vee^{\mathfrak{E}} (b \vee^{\mathfrak{E}} c)) = (3 : 3 : 3 : 3 : 1 : 1 : 1). \tag{3.8}$$

Consider the following complementary cases:

- $(J_3 \cup J_2) = \varnothing$.
  Then, by (3.8), $E \ni d = (I \times \{1\})$, while $I \supseteq J_0 \neq \varnothing$, whereas $\mathfrak{A}_4$, having no proper subalgebra other than $\mathfrak{A}_{4:2}$, is generated by any element of $(4 \setminus A_{4:2}) = \{1, 2\} \ni 1$, in which case $\{\langle l, I \times \{l\}\rangle \mid l \in 4\}$ is an embedding of $\mathcal{A}_4$ into $\mathcal{E} \in \mathrm{Mod}(C)$, and so $C \subseteq C_4 \subseteq C_4^{\mathrm{NP}}$.
- $(J_3 \cup J_2) \neq \varnothing$.
  Note that $\mathcal{A}_4 \times \mathcal{A}_{4:2}$ is generated by $\langle 1, 3\rangle$, for ${\sim}^{\mathfrak{A}_4^2}\langle 1, 3\rangle = \langle 2, 0\rangle$, while $(\langle 1, 3\rangle (\wedge|\vee)^{\mathfrak{A}_4^2} \langle 2, 0\rangle) = \langle 1|2, 0|3\rangle$, whereas $\partial_1^{\mathfrak{A}_4^2}\langle 1|2, 0/3\rangle = \langle 0|3, 0/3\rangle$. In that case, since $J_0 \neq \varnothing$, by (3.8), we see that $\{\langle\langle l, m\rangle, (m : m : m : m : l : l : l)\rangle \mid l \in 4, m \in A_{4:2}\}$ is an embedding of $\mathcal{A}_4 \times \mathcal{A}_{4:2}$ into $\mathcal{E} \in \mathrm{Mod}(C)$, and so, by Theorem 3.11, $C \subseteq C_4^{\mathrm{NP}}$.

After all, combining Theorems 3.3, 3.11, Corollary 3.13, Lemma 3.14 and (3.5), we eventually get the following four-valued analogue of Theorem 4.13 of [15]:

THEOREM 3.15. *Proper consistent extensions of C_4 form the two-element chain $C_4^{\mathrm{NP}} = \mathrm{Cn}_{\mathcal{A}_4 \times \mathcal{A}_{4:2}} \subsetneq C_4^{\mathrm{MP/R}} = C_4^{\mathrm{PC}}$.*

Concluding this discussion, we should like to highlight three more negative consequences of (3.7), aside from the second sentence of Lemma 3.12. First of all, if any $\varepsilon \subseteq \mathrm{Fm}_{\Sigma_4}^2$ with at most one variable x_0 defined truth in $\mathcal{A}_4$, then, as $1 \in D^{\mathcal{A}_4}$, we would have $\mathfrak{A}_4 \models (\bigwedge \varepsilon)[x_0/1]$, in which case, since $\partial_1^{\mathfrak{A}_4}1 = 0$, by (3.7), we would get $\mathfrak{A}_4 \models (\bigwedge \varepsilon)[x_0/0]$, and so would eventually get $0 \in D^{\mathcal{A}_4}$. In particular, as opposed to $C_3 = LP$, the generic technique developed in §3 of [15] is not applicable to C_4, while there is no secondary unary connective $\Diamond$ satisfying (3.9) below, for, otherwise, $\{\Diamond x_0 \approx \top\}$ would define truth in $\mathcal{A}_4$, whereas, by Propositions 6 and 7 of [18] as well as Lemmas 3.2 of [17] and 3.1, C_4 is not algebraizable. Likewise, if any $\varepsilon \subseteq \mathrm{Fm}^2_{\Sigma_4}$ with at most two variables x_0 and x_1 was an equational implication for $\mathcal{A}_4$, then, as $1 \in D^{\mathcal{A}_4}$, we would have $\mathfrak{A}_4 \models (\bigwedge \varepsilon)[x_0/2, x_1/1]$, in which case, since $\partial_1^{\mathfrak{A}_4}(2|1) = (3|0)$, by (3.7), we would get $\mathfrak{A}_4 \models (\bigwedge \varepsilon)[x_0/3, x_1/0]$, and so would eventually get $0 \in D^{\mathcal{A}_4}$, for $3 \in D^{\mathcal{A}_4}$. And what is more, by (3.7), $h \triangleq \partial_1^{\mathfrak{A}_4}$ is not injective, for $|A_4| = 4 \not\leq 2 = |A_{4:2}|$, in which case, as $|A_4| = 4 \neq 1$, $(\ker h) \in (\mathrm{Con}(\mathfrak{A}_4) \setminus \{A_4^2\})$ is not diagonal, and so $\mathfrak{A}_4$ has no ternary discriminator. This is why the advanced algebraic methods used in the next subsubsection are not applicable to the four-valued case.

3.1.2. *The more-than-four-valued case.* Here, it is supposed that $n > 4$, in which case $2 \in N_n$ (in particular, $\nabla_2 \in \Sigma_n$), and so we have the secondary unary connective $\Diamond x_0 \triangleq \partial_1 \nabla_2 x_0$. Then,

$$\Diamond^{\mathfrak{A}_n} a \triangleq \begin{cases} 0 & \text{if } a = 0, \\ n-1 & \text{otherwise}, \end{cases} \tag{3.9}$$

for all $a \in n$, in which case

$$(\mathrm{img}\, \Diamond^{\mathfrak{A}_n}) \subseteq A_{n:2}, \tag{3.10}$$

and so $\mathcal{A}_n$ is $\supset$-implicative (in particular, C_n is so), where $(x_0 \supset x_1) \triangleq (\sim\Diamond x_0 \vee \Diamond x_1)$, for $\mathcal{A}_{n:2}$ is both $\sim$-negative and $\vee$-disjunctive. Set $(x_0 \equiv x_1) \triangleq ((x_0 \supset x_1) \wedge (x_1 \supset x_0))$. Then, as $\mathcal{A}_n$ is both $\supset$-implicative and $\wedge$-conjunctive, by (3.10), we have:

$$(a \equiv^{\mathfrak{A}_4} b) = \begin{cases} n-1 & \text{if } (a = 0) \Leftrightarrow (b = 0), \\ 0 & \text{otherwise}, \end{cases} \tag{3.11}$$

for all $a, b \in n$.

Further, we have the one more secondary binary connective of Σ_n: $(x_0 \equiv_n x_1) \triangleq (\wedge \langle \langle \partial_{k+1} x_0 \equiv \partial_{k+1} x_1 \rangle_{k \in m_n}, x_0 \equiv x_1 \rangle)$. Finally, put $(x_0 \leftrightarrow x_1) \triangleq ((x_0 \equiv_n x_1) \wedge (\sim x_0 \equiv_n \sim x_1))$. Then, by Lemma 3.1, (3.11) and the $\wedge$-conjunctivity of $\mathcal{A}_{n:2}$, we eventually get:

$$(a \leftrightarrow^{\mathfrak{A}_4} b) = \begin{cases} n-1 & \text{if } a = b, \\ 0 & \text{otherwise}, \end{cases} \tag{3.12}$$

for all $a, b \in n$. By (3.9), we first conclude that $\varepsilon_n \triangleq \{\Diamond x_0 \approx \top\}$ defines truth in $\mathcal{A}_n$, in which case, in particular, by (3.12), C_n is equivalent to the quasivariety generated by $\mathfrak{A}_n$ with respect to $\leftrightarrow$ and ε_n in the sense of [13], and so is algebraizable with respect to $\leftrightarrow$ and ε_n, in view of Proposition 6 of [18]. And what is more, by (3.12), we see that $((x_0 \leftrightarrow x_1) \wedge x_2) \vee (\sim(x_0 \leftrightarrow x_1) \wedge x_0)$ is a ternary discriminator for $\mathfrak{A}_n$. In this way, combining Theorem 3.3 of [15] as well as Corollaries 4.3, 4.6, 4.12 and Example B.2 of [19] with (3.5) and Proposition 3.9, we eventually get:

Theorem 3.16. *Proper consistent extensions of C_n form the two-element chain $C_n^{\mathrm{NP}} = \mathrm{Cn}_{\mathcal{A}_n \times \mathcal{A}_{n:2}} \subsetneq C_n^{\mathrm{MP/R}} = C_n^{\mathrm{PC}}$. Moreover, C_n^{PC} is the axiomatic extension of C_n relatively axiomatized by* (3.6) *with $\diamond = \supset$, in which case $C_n^{\mathrm{PC}}(\varnothing) \neq C_n(\varnothing)$.*

This subsumes Theorem 3.3, yielding another insight into it, and resembles the corresponding results obtained in [16]/[19] for HZ/LA [4]/[1], respectively. In general, combining Theorems 4.13 of [15], 3.15, 3.16 as well as Lemmas 4.14 of [15] and 3.12 with (3.5) and Proposition 3.8, we have the following universal result:

Corollary 3.17. *Let $n > 2$. Then, proper consistent extensions of C_n form the two-element chain $C_n^{\mathrm{NP}} = \mathrm{Cn}_{\mathcal{A}_n \times \mathcal{A}_{n:2}} \subsetneq C_n^{\mathrm{MP/R}} = C_n^{\mathrm{PC}}$. Moreover, $C_n^{\mathrm{NP}}(\varnothing) = C_n(\varnothing)$, in which case C_n^{NP} is not an axiomatic extension of C_n. And what is more, providing $n \leqslant | > 4$, $C_n^{\mathrm{PC}}(\varnothing) = | \neq C_n(\varnothing)$, while C_n^{PC} "is not an"|"is the" axiomatic extension of C_n |"relatively axiomatized by* (3.6) *with $\diamond = \supset$", respectively.*

This does not depend upon whether *exactly* $n = 3$, and so definitely unifies $C_3 = LP$ with its minimally more-than-three-valued maximally $\sim$-paraconsistent expansions.

3.2. Sequent calculi. Here, we propose Cut-free sequent axiomatizations of the introduced logics tacitly using Lemma 3.1 and entirely following the generic approach elaborated in [17] but naturally using the variables x_0 and x_i, where $i \in \omega$, instead of p and p_{i+1}, respectively, for *both* p and p_1 occur in no sequent [rule] actually dealt with here, and involving all substitutional instances of sequent rules in constructing derivations as well as at once endowing the calculi to be constructed with structural rules other than Cut and Contraction, and so disregarding the item (i) of Definition 1 therein and taking merely those of context-free *canonical* sequent axioms (i.e., ones with disjoint injective left and right sides — viz., components, constituted by formulas in $\Im_n$ alone and ordered according to any total ordering of $\Im_n$ [e.g., the one given by $x_0 < \{\partial_k\langle\sim\rangle x_0 < \partial_l\langle\sim\rangle x_0 < (\partial_m)\}{\sim}x_0$ {for all $k, l(, m) \in N_{n-}$ such that $k < l\langle(<)\rangle m$} and supposed below]), in the item (ii) of Deinition 1 therein (i.e., true in $\mathcal{A}_n$), which are minimal under subsumption *partial* (because, for all Σ_n-formulas η and ζ with at most one variable x_0, $\eta = x_0 = \zeta$, whenever $\eta(\zeta) = x_0$) ordering between canonical sequents under their identification with their semantically treating disjuncts (cf. [23] for the definition of the *subsumption quasi*-ordering between disjuncts), that are proved right below to be as follows:

$$\vdash [\partial_i]x_0, \{\partial_j\}{\sim}x_0 \qquad [i \in N_{n-}]\{j \in N_{n-}\} \tag{3.13}$$

$$\partial_i x_0 \vdash [\partial_j]x_0 \qquad [N_{n-} \ni j <]i \in N_{n-} \tag{3.14}$$

$$\partial_{m_n} x_0, \partial_{m_n}{\sim}x_0 \vdash \qquad n > 3 \text{ is even.} \tag{3.15}$$

PROOF. Consider any $k \in n$. First, we prove that (3.15) is true in $\mathcal{A}_n$ under $[x_0/k]$, by contradiction. For suppose it is not true in $\mathcal{A}_n$ under $[x_0/k]$, in which case $k > m_n < (n-1-k)$, and so $k \notin N_{n-} \not\ni (n-1-k)$. Then, $(n-1-k) \in N_n \ni k$, in which case $(n-1-k) = k$, and so $n = ((2 \cdot k) + 1)$ is odd. This contradiction shows that (3.15) is true in $\mathcal{A}_n$. Next, for proving the truth of (3.13) in $\mathcal{A}_n$ under $[x_0/k]$, assume $k \leqslant (0[+i])$. Then, in case $k = 0$, we have $\{\partial_j^{\mathfrak{A}_n}\}{\sim}^{\mathfrak{A}_n}k = (n-1) \neq 0$ [while, otherwise, ${\sim}^{\mathfrak{A}_n}k \geqslant {\sim}^{\mathfrak{A}_n}i \geqslant {\sim}^{\mathfrak{A}_n}m_n > m_n \geqslant (0\{+j\})$, in which case $\{\partial_j^{\mathfrak{A}_n}\}{\sim}^{\mathfrak{A}_n}k \neq 0$], and so (3.13) is true in $\mathcal{A}_n$. Likewise, for proving the truth of (3.14) in $\mathcal{A}_n$ under $[x_0/k]$, assume $k \leqslant (0[+j]) \leqslant i$, in which case $\partial_i^{\mathfrak{A}_n}k = 0$, and so (3.14) is true in $\mathcal{A}_n$. In particular, any minimal canonical sequent true in $\mathcal{A}_n$ is *strictly canonical* in the sense that, if the left/right side of it contains $(\partial_i[{\sim}]x_0)/(\{\partial_i\}[{\sim}]x_0)$, where $i \in N_{n-}$, then this does not contain $(\{\partial_j\}[{\sim}]x_0)/(\partial_j[{\sim}]x_0)$, where $j \in N_{n-}$ {and $j > / < i$}, respectively. On the other hand, those of strictly canonical sequents, which are subsumed by neither (3.13) nor (3.14) nor (3.15), subsume either of the following sequents, each of which is proved not true in $\mathcal{A}_n$, and so is each of the former ones:

1. $x_0, {\sim}x_0 \vdash$.
   This is not true in $\mathcal{A}_n$ under $[x_0/1]$.
2. $[\partial_i]{\sim}^m x_0 \vdash \{\partial_j\}{\sim}^{1-m}x_0$, where $m \in 2$ [and $i \in N_{n-}$] {as well as $j \in N_{n-}$}.
   This is not true in $\mathcal{A}_n$ under $[x_0/({\sim}^{\mathfrak{A}_n})^m(n-1)]$.
3. $\partial_i x_0, \partial_j{\sim}x_0 \vdash$, where $i, j \in N_{n-}$, while n is odd,
   in which case $N_{n-} \neq \varnothing$, and so $n > 3$, while $(n-1) < n$ is even, and so $l \triangleq \frac{n-1}{2} \in n$. Then, $i < l > j$ and ${\sim}^{\mathfrak{A}_n}l = l$, in which case the sequent under consideration is not true in $\mathcal{A}_n$ under $[x_0/l]$.
4. $[\partial_i]x_0, \{\partial_j\}{\sim}x_0 \vdash ((\partial_\imath x_0)|(\partial_\jmath{\sim}x_0))$, where $(\imath|\jmath)[,i]\{,j\} \in N_{n-}$ [and $i < \imath$] {as well as $j < \jmath$},
   in which case $0 < (\imath|\jmath) \leqslant m_n$, and so $(0\{+j\}|0[+i]) \leqslant m_n < {\sim}^{\mathfrak{A}_n}m_n \leqslant {\sim}^{\mathfrak{A}_n}(\imath|\jmath)$. In this way, the sequent under consideration is not true in $\mathcal{A}_n$ under $[x_0/(\imath|{\sim}^{\mathfrak{A}_n}\jmath)]$. [{In particular, when taking $(\imath|\jmath) = m_n$, in case $(i|j) \neq m_n$, we see that $\partial_i x_0, \partial_j{\sim}x_0 \vdash$ is not true in $\mathcal{A}_n$.}]

Finally, by (4), we eventually conclude that both (3.13) and (3.14) and (3.15) has no proper (viz., non-equal) subsequent [under inclusion of the images (viz., contents) of the left and right sides pairwise] true in $\mathcal{A}_n$, in which case it, being strictly canonical, is minimal under subsumption among strictly canonical sequents true in $\mathcal{A}_n$, because, for any Σ_n-formula φ with at most one variable x_0 and any strictly canonical sequent Φ, $\varphi = x_0$, whenever $\Phi[x_0/\varphi]$ is equal to either of the three sequents under consideration, and so among canonical sequents true in $\mathcal{A}_n$. ⊣

Likewise, in case $n > 3$, it suffices to take solely context-free sequent axioms in the items (iii-iv) of Definition 1 of [17] that are clearly as follows (where $i \in N_{n-}$):

$$\vdash (\partial_i)\top, \tag{3.16}$$

$$(\partial_i){\sim}\top \vdash, \tag{3.17}$$

$$\vdash (\partial_i){\sim}\bot, \tag{3.18}$$

$$(\partial_i)\bot \vdash . \tag{3.19}$$

Further, the only $(\Im_n, \Sigma_n)$-type[s], not being $\Im_n$-complex, is [resp., are] $(\sim)$ [as well as both (∂_i) and $\partial_i(\sim)$, where $i \in N_{n-}$]. Then, we have the following Σ_n-sequential $\Im_n$-table $\mathcal{T} = \langle \lambda_{\mathcal{T}}, \rho_{\mathcal{T}} \rangle$ of rank $(0,0)$ for $\mathcal{A}_n$ yielding the rules in the item (v) of Definition 1 of [17], constituting collectively with both the axioms (3.13), (3.14), (3.15) [as well as (3.16), (3.17), (3.18) and (3.19), whenever $n > 3$] and structural rules but Cut and Contraction the resulting Cut-free Gentzen-style axiomatization $\mathcal{S}_n$ of C_n with admissible Cut and Contraction. First, for all $i \in N_{n-}$, $\partial_i^{\mathfrak{A}_n} \in \hom(\mathfrak{A}_n{\restriction}\Sigma_+, \mathfrak{A}_{n:2}{\restriction}\Sigma_+)$. Moreover, $D^{\mathcal{A}_n}$ is a prime filter of $\mathfrak{A}_n$, while the identities (2.1), (2.2) and (2.3) are true in the Kleene lattice $(\mathfrak{A}_n{\restriction}\Sigma_\sim) = \mathfrak{K}_n$. Therefore, [for all $i \in N_{n-}$] one can naturally choose:

$$\begin{aligned}
\lambda_{\mathcal{T}}([\partial_i](\wedge)) &\triangleq \{[\partial_i]x_0, [\partial_i]x_1 \vdash\},\\
\lambda_{\mathcal{T}}([\partial_i]{\sim}(\vee)) &\triangleq \{[\partial_i]{\sim}x_0, [\partial_i]{\sim}x_1 \vdash\},\\
\lambda_{\mathcal{T}}([\partial_i](\vee)) &\triangleq \{[\partial_i]x_0 \vdash, [\partial_i]x_1 \vdash\},\\
\lambda_{\mathcal{T}}([\partial_i]{\sim}(\wedge)) &\triangleq \{[\partial_i]{\sim}x_0 \vdash, [\partial_i]{\sim}x_1 \vdash\},\\
\rho_{\mathcal{T}}([\partial_i](\vee)) &\triangleq \{\vdash [\partial_i]x_0, [\partial_i]x_1\},\\
\rho_{\mathcal{T}}([\partial_i]{\sim}(\wedge)) &\triangleq \{\vdash [\partial_i]{\sim}x_0, [\partial_i]{\sim}x_1\},\\
\rho_{\mathcal{T}}([\partial_i](\wedge)) &\triangleq \{\vdash [\partial_i]x_0, \vdash [\partial_i]x_1\},\\
\rho_{\mathcal{T}}([\partial_i]{\sim}(\vee)) &\triangleq \{\vdash [\partial_i]{\sim}x_0, \vdash [\partial_i]{\sim}x_1\},\\
\lambda_{\mathcal{T}}([\partial_i]{\sim}(\sim)) &\triangleq \{[\partial_i]x_0 \vdash\},\\
\rho_{\mathcal{T}}([\partial_i]{\sim}(\sim)) &\triangleq \{\vdash [\partial_i]x_0\}.
\end{aligned}$$

And what is more, for all $i \in N_{n-}$, we have $(\operatorname{img} \partial_i^{\mathfrak{A}_n}) \subseteq A_{n:2}$, while $\partial_i^{\mathfrak{A}_n} l = l$, for all $l \in A_{n:2}$, and so, by the $\sim$-negativity of $\mathcal{A}_{n:2}$, one can choose, for all $\imath \in N_{n-}$:

$$\begin{aligned}
\lambda_{\mathcal{T}}(\partial_i(\partial_\imath)) &\triangleq \{\partial_\imath x_0 \vdash\},\\
\rho_{\mathcal{T}}(\partial_i(\partial_\imath)) &\triangleq \{\vdash \partial_\imath x_0\},\\
\lambda_{\mathcal{T}}([\partial_i]{\sim}(\partial_\imath)) &\triangleq \{\vdash \partial_\imath x_0\},\\
\rho_{\mathcal{T}}([\partial_i]{\sim}(\partial_\imath)) &\triangleq \{\partial_\imath x_0 \vdash\}.
\end{aligned}$$

Likewise, in case $n > 4$, for all $j \in N_n$, we have $(k = l) \Leftrightarrow (\nabla_j^{\mathfrak{A}_n} k = l)$, for all $k \in n$, and so one can choose:

$$\begin{aligned}
\lambda_{\mathcal{T}}([\sim](\nabla_j)) &\triangleq \{[\sim]x_0 \vdash\},\\
\rho_{\mathcal{T}}([\sim](\nabla_j)) &\triangleq \{\vdash [\sim]x_0\}.
\end{aligned}$$

Finally, in that case, for all $m \in 2$, we have $\partial_i^{\mathfrak{A}_n}(\sim^{\mathfrak{A}_n})^m \nabla_j^{\mathfrak{A}_n} k = 0$ iff either $(\sim^{\mathfrak{A}_n})^m k = 0$ or both $(\sim^{\mathfrak{A}_n})^{1-m} k \neq 0$ and $(n-1-)^m j \leqslant i$, and so one can eventually choose:

$$\begin{aligned}
\lambda_{\mathcal{T}}(\partial_i{\sim}^m(\nabla_j)) &\triangleq (\{\sim^m x_0 \vdash \mid i < (n-1-)^m j\}\\
&\quad \cup \{\sim^m x_0 \vdash \sim^{1-m} x_0\}),\\
\rho_{\mathcal{T}}(\partial_i{\sim}^m(\nabla_j)) &\triangleq (\{\vdash \sim^m x_0\}\\
&\quad \cup \{\sim^{1-m} x_0 \vdash \mid i < (n-1-)^m j\}).
\end{aligned}$$

It is remarkable that, in case $n = 3$, the resulting sequent calculus $\mathcal{S}_n$ is the exactly already-known (due to [17]) one resulted from that discovered in [11] for Belnap's four-valued logic B_4 [3] by adding the *Excluded Middle Law* sequent axiom (3.13) alone — $\vdash x_0, \sim x_0$, for $N_{n-} = \varnothing$, in that case.

Finally, in case $n > 4$, by (3.9), we see that $\{\Diamond x_0 \lessapprox \Diamond x_1\}$ is an equational implication for $\mathcal{A}_n$. On the other hand, by Remark 1 of [18], $\mathcal{A}_3$ has an equational implication. In this way, taking the fourth sentence of the last paragraph of Subsubsection 3.1.1 into account, we conclude that $\mathcal{A}_n$ has an equational implication (that is, the sequent calculus $\widetilde{\mathcal{S}}_n$ resulted from the constructed one $\mathcal{S}_n$ by adding Cut and Contraction is algebraizable; cf. Theorems 10 and 13 of [18]) iff $n \neq 4$, in which case, in view of (16) of [18], the sequent calculus $\widetilde{\mathcal{S}}_n$ is equivalent to the quasivariety generated by $\mathfrak{A}_n$ in the sense of [13]. On the other hand, by Lemma 5 of [18], the constructed calculus $\mathcal{S}_n$ as such is not algebraizable. And what is more, by Theorem 7 of [18] as well as Lemmas 3.2 of [17] and 3.1, $C_3 = LP$ is not algebraizable, for $K'_4 \triangleq \{\langle 2,2\rangle, \langle 1,2\rangle, \langle 1,0\rangle, \langle 0,0\rangle\}$ forms a subalgebra of $\mathfrak{A}_3^2$, while $(\pi_1 \upharpoonright K'_4) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}_3^2 \upharpoonright K'_4, \mathcal{A}_{3:2})$ is not injective, in which case $\mathcal{A}_3^2 \upharpoonright K'_4$ is not simple. Thus, taking the third sentence of the last paragraph of Subsubsection 3.1.1 and the seventh sentence of Subsubsection 3.1.2 into account, we see that C_n is algebraizable iff $n > 4$, in which case it is equivalent to the quasivariety generated by $\mathfrak{A}_n$, and so to $\widetilde{\mathcal{S}}_n$ in the sense of [13].

3.3. Hilbert-style calculi. First of all, let us summarize how the universal approach developed in [21] is applicable to C_n, when $n > (2/4)$, in which case $\mathcal{A}_n$ is $\vee$-disjunctive|"$\supset$-implicative (cf. Subsubsection 3.1.2)", upon the basis of Lemma 3.1 and Subsection 3.2. Let $\mathcal{A}$ be the finite set of Σ_n-sequents (3.13), (3.14) and (3.15) [as well as (3.16), (3.17), (3.18) and (3.19), whenever $n > 3$] collectively with the following supplementary ones: for each $\Im_n$-complex $(\Im_n, \Sigma_n)$-type $\iota(F)$, where $F \in ((\Sigma_n \setminus \{\vee\}|\Sigma_n))$ of arity $l > 0$, all sequents resulted from those in $(\lambda/\rho)_{\mathcal{T}}(\iota(F))$ by adding $\iota(F(x_0, \ldots, x_{l-1}))$ to their right/left sides, respectively. (Note that, in case $\iota = [\partial_i]{\sim}x_0$ and $F = \partial_\imath$, where $\imath[, i] \in N_{n-}$, the resulting sequent is subsumed by (3.13), and so can be omitted.) Then, let $\sigma_{+1} \triangleq [x_i/x_{1+1}]_{i\in\omega}$, in which case we get the finite set $\mathcal{B} \triangleq \{((\bar{\phi} \circ (\vee x_0)) \vdash \langle\bar{\psi}, x_0\rangle)|(\varnothing \vdash ((\psi/x)_0, \bar{\phi}/, (\bar{\psi} \circ (\supset x_0)))) \mid k \in \omega \ni m| = / \neq 1, \bar{\phi} \in \mathrm{Fm}^k_{\Sigma_n}, \bar{\psi} \in \mathrm{Fm}^m_{\Sigma_n}, (\bar{\phi} \vdash \bar{\psi}) \in (\sigma_{+1}[\mathcal{A}]|(\mathcal{A}/\sigma_{+1}[\mathcal{A}]))\}$ of Σ_n-sequents with non-empty right sides|" and empty left ones", and so eventually get the finite axiomatization $\mathcal{H}^{\vee|\supset}_n$ of C_n resulted from any finite axiomatization of the $(\vee|\supset)$-fragment of the classical logic by adding the following Σ_n-rules|-axioms: for each $(\bar{\phi} \vdash \bar{\psi}) \in \mathcal{B}$, where $\bar{\phi} \in \mathrm{Fm}^k_{\Sigma_n}$ and $\bar{\psi} \in \mathrm{Fm}^m_{\Sigma_n}$, while $k, m \in \omega$, whereas $m \neq 0| = k$, the Σ_n-rule|-axiom $(\mathrm{img}\,\bar{\phi}) \to ((\vee|\subset)\bar{\psi})$, respectively. And what is more, due to rules arising from both $(\lambda/\rho)_{\mathcal{T}}([\partial_i](\wedge))$ and $(\lambda/\rho)_{\mathcal{T}}((\partial_i|[\partial_i])(\vee))$, where $i \in N_{n-}$, as well as those satisfied in the $(\vee|\supset)$-fragment of the classical logic, some rules of the resulting calculus are then subject to evident equivalent transformations, tacitly made below.

In this way, a definitive version of $\mathcal{H}^{\vee}_n$ is constituted by the following axioms and rules, where $i, \imath \in N_{n-}$, $j \in N_n$ and $m \in 2$:

$$
\begin{gathered}
[\partial_i]x_0 \vee \{\partial_\imath\}{\sim}x_0 \\
(\partial_i)\top \quad (\partial_i){\sim}\bot \qquad n > 3 \\
\frac{(\partial_i){\sim}\top \vee x_0}{x_0} \quad \frac{(\partial_i)\bot \vee x_0}{x_0} \qquad n > 3 \\
\frac{\partial_i x_1 \vee x_0}{[\partial_\imath]x_1 \vee x_0} \qquad [\imath <]i \\
\frac{(\partial_{m_n} x_1 \wedge \partial_{m_n}{\sim}x_1) \vee x_0}{x_0} \qquad n > 3 \text{ is even} \\
\frac{(\partial_i{\sim}^m\nabla_j x_1 \wedge {\sim}^{1-m}x_1) \vee x_0}{x_0} \quad \frac{{\sim}^m x_1 \vee x_0}{\partial_i{\sim}^m\nabla_j x_1 \vee x_0} \quad n > 4, i < (n-1-)^m j \\
\frac{{\sim}^m x_1 \vee x_0}{({\sim}^{1-m}x_1 \vee \partial_i{\sim}^m\nabla_j x_1) \vee x_0} \quad \frac{\partial_i{\sim}^m\nabla_j x_1 \vee x_0}{{\sim}^m x_1 \vee x_0} \quad \frac{{\sim}^m\nabla_j x_1 \vee x_0}{{\sim}^m x_1 \vee x_0}\updownarrow \quad n > 4
\end{gathered}
\tag{3.20}
$$

$$\frac{\partial_i\partial_\imath x_1 \vee x_0}{\partial_\imath x_1 \vee x_0}\updownarrow \qquad \frac{(\partial_\imath x_1 \wedge [\partial_i]{\sim}\partial_\imath x_1) \vee x_0}{x_0}$$

$$\frac{x_0 \vee x_0}{x_0} \qquad \frac{x_0}{x_0 \vee x_1} \qquad \frac{(x_0 \vee x_1) \vee x_2}{(x_1 \vee x_0) \vee x_2} \qquad \frac{(x_0 \vee (x_1 \vee x_2)) \vee x_3}{((x_0 \vee x_1) \vee x_2) \vee x_3}$$

$$\frac{[\partial_i](x_1 \wedge x_2) \vee x_0}{[\partial_i]x_1 \vee x_0} \quad \frac{[\partial_i](x_1 \wedge x_2) \vee x_0}{[\partial_i]x_2 \vee x_0} \quad \frac{[\partial_i]x_1 \vee x_0 \quad [\partial_i]x_2 \vee x_0}{[\partial_i](x_1 \wedge x_2) \vee x_0}$$

$$\frac{(\partial_i x_1 \vee \partial_i x_2) \vee x_0}{\partial_i(x_1 \vee x_2) \vee x_0}\updownarrow \qquad \frac{[\partial_i]x_1 \vee x_0}{[\partial_i]{\sim\sim}x_1 \vee x_0}\updownarrow$$

$$\frac{([\partial_i]{\sim}x_1 \wedge [\partial_i]{\sim}x_2) \vee x_0}{[\partial_i]{\sim}(x_1 \vee x_2) \vee x_0}\updownarrow \qquad \frac{([\partial_i]{\sim}x_1 \vee [\partial_i]{\sim}x_2) \vee x_0}{[\partial_i]{\sim}(x_1 \wedge x_2) \vee x_0}\updownarrow$$

It is remarkable that, in case $n = 3$, $\mathcal{H}_n^{\vee}$ is exactly the axiomatization of $C_n = LP$ found in [11] (cf. Corollary 5.3 therein) under enhancement of that of B_4 being due to [21].

Likewise, a definitive version of $\mathcal{H}_n^{\supset}$ is constituted by (2.7), (2.9), (2.10) and (2.11) with $\diamond = \supset$ as well as by both (3.20) and the following axioms, where $i, \imath \in N_{n-}$, $j \in N_n$ and $m \in 2$:

$$(\partial_i){\sim}\top \supset x_0$$
$$(\partial_i)\bot \supset x_0$$
$$[\partial_i](x_0 \wedge x_1) \supset [\partial_i]x_m$$
$$[\partial_i]x_0 \supset ([\partial_i]x_1 \supset [\partial_i](x_0 \wedge x_1))$$
$$[\partial_i]x_m \supset [\partial_i](x_0 \vee x_1)$$
$$([\partial_i]x_1 \supset x_0) \supset (([\partial_i]x_2 \supset x_0) \supset ([\partial_i](x_1 \vee x_2) \supset x_0))$$
$$\partial_i x_0 \supset [\partial_\imath]x_0 \qquad [\imath <]i$$
$$\partial_{m_n} x_1 \supset (\partial_{m_n}{\sim}x_1 \supset x_0) \qquad n \text{ is even}$$
$${\sim}^{1-m}x_1 \supset (\partial_i{\sim}\nabla_j x_1 \supset x_0) \qquad i < (n-1-)^m j$$
$${\sim}^m x_0 \supset \partial_i{\sim}\nabla_j x_0 \qquad i < (n-1-)^m j$$
$${\sim}^m x_0 \supset ({\sim}^{1-m}x_0 \vee \partial_i{\sim}\nabla_j x_0)$$
$$\partial_i{\sim}\nabla_j x_0 \supset {\sim}^m x_0$$
$$\partial_\imath x_1 \supset ([\partial_i]{\sim}\partial_\imath x_1 \supset x_0)$$
$${\sim}^m x_0 \equiv {\sim}^m \nabla_j x_0$$
$$\partial_i\partial_\imath x_0 \equiv \partial_\imath x_0$$
$$[\partial_i]{\sim\sim}x_0 \equiv [\partial_i]x_0$$
$$[\partial_i]{\sim}(x_0 \wedge x_1) \equiv ([\partial_i]{\sim}x_0 \vee [\partial_i]{\sim}x_1)$$
$$[\partial_i]{\sim}(x_0 \vee x_1) \equiv ([\partial_i]{\sim}x_0 \wedge [\partial_i]{\sim}x_1)$$

3.4. Equivalent varieties. In case $n = 3$, the quasivariety generated by $\mathfrak{A}_n = \mathfrak{K}_3$ and equivalent to $\widetilde{\mathcal{S}}_n$ is well-known to be the variety of Kleene lattices (cf., e.g., Proposition 3.4 of [14]). Here, we mainly explore the opposite case, when $n > 4$, in which case both C_n and $\widetilde{\mathcal{S}}_n$ are equivalent to the quuasi-variety $\mathbf{QV}(\mathfrak{A}_n)$ generated by $\mathfrak{A}_n$, and so Corollary 3.24 of [13] yields a finite *quasi*-equational axiomatization of $\mathbf{QV}(\mathfrak{A}_n)$ upon the basis of $\mathcal{H}_n^{\supset}$. However, such an axiomatization would be too cumbersome as well as far from being algebraically (more specifically, lattice-theoretically) natural. And what is more, since $\mathfrak{A}_n$ has a ternary discriminator, in view of Theorems 1.3, 2.5 (collectively with the comment 4 after it) and 2.6 of [9], $\mathbf{QV}(\mathfrak{A}_n)$ is a variety. On the other hand, the above axiomatization contains two non-equational quasi-identities because of the items (i-ii) of Corollary 3.24 of [13], for, in particular, (2.7) is not an axiom. Nevertheless, we find below *equational algebraically natural* finite axiomatizations of $\mathbf{QV}(\mathfrak{A}_n)$.

Given any $i \in n$, take any enumeration $\overline{\Im}_{n,\mathcal{A}_n,i,+/-}$ of $\Im_{n,\mathcal{A}_n,i,+/-}$, respectively (cf. Lemma 3.1). Then, put $\tau_i \triangleq (\Diamond(\wedge\langle\overline{\Im}_{n,\mathcal{A}_n,i,+},\top\rangle) \wedge {\sim}\Diamond(\vee\langle\overline{\Im}_{n,\mathcal{A}_n,i,-},\bot\rangle)$, in which case, by the $\wedge$-conjunctivity and $\vee$-disjunctivity of $\mathcal{A}_n$ as well as (3.9), $\tau_i^{\mathfrak{A}_n}(i) = (n-1)$, and so the following identity is satisfied in $\mathfrak{A}_n$:

$$(\vee\langle\tau_i\rangle_{i\in n}) \approx \top. \tag{3.21}$$

Now, consider any $j \in n$ distinct from i. Then, by Lemma 3.1, there is either some $\iota \in \Im_{n,\mathcal{A}_n,i,+}$ such that $\iota^{\mathfrak{A}_n}(j) \notin D^{\mathcal{A}_n}$ or some $\iota \in \Im_{n,\mathcal{A}_n,i,-}$ such that $\iota^{\mathfrak{A}_n}(j) \in D^{\mathcal{A}_n}$, in which case, by the $\wedge$-conjunctivity and $\vee$-disjunctivity of $\mathcal{A}_n$ as well as (3.9), $\tau_i^{\mathfrak{A}_n}(j) = 0$, and so the following identities are satisfied in $\mathfrak{A}_n$:

$$(\tau_i \wedge \tau_j) \approx \bot \qquad n \ni i \neq j \in n \tag{3.22}$$

$$(\wedge\langle\langle\tau_{k_l}(x_l)\rangle_{l\in m},\top\rangle) \lessapprox \tau_k(F(x_0,\dots,x_{m-1})), \tag{3.23}$$

where $F \in (\Sigma_n \setminus \{\vee,\top\})$ of arity $m \in \omega$, $\bar{k} \in n^m$ and $k = F^{\mathfrak{A}_n}(\bar{k})$. Likewise, by (3.11) and (3.12), the following identity is then also satisfied in $\mathfrak{A}_n$:

$$(\wedge\langle\tau_i(x_0) \equiv \tau_i(x_1)\rangle_{i\in n}) \lessapprox (x_0 \leftrightarrow x_1). \tag{3.24}$$

Next, an *n-graded Kleene algebra* is any Σ_n-algebra, the $\Sigma_{\sim,01}$-reduct of which is a Kleene algebra and which satisfies the identities (3.21), (3.22), (3.23), (3.24) and the following additional ones:

$$(x_0 \wedge (x_0 \leftrightarrow x_1)) \lessapprox x_1, \tag{3.25}$$

$$\left.\begin{array}{l}\tau_i \lessapprox (x_1 \supset \tau_i),\\ (\tau_i \vee (\tau_i \supset x_1)) \approx \top,\end{array}\right. \qquad i \in n, \tag{3.26}$$

the variety of all them being denoted by GKA_n.

THEOREM 3.18. *$\mathfrak{A}_n$ is an n-graded Kleene algebra. Conversely, any [finite] n-graded Kleene algebra is embeddable into a [finite] direct power of $\mathfrak{A}_n$. In particular,* $\mathsf{GKA}_n = \mathbf{QV}(\mathfrak{A}_n)$.

PROOF. First, $(\mathfrak{A}_n{\restriction}\Sigma_{\sim,01}) = \mathfrak{K}_{n,01}$ is a Kleene algebra. Next, the fact that $\mathfrak{A}_n$ satisfies (3.25) is by (3.12). Further, the fact that, for each $i \in n$, $\mathfrak{A}_n$ satisfies (3.26) is by the fact that $(\mathrm{img}\,\tau_i^{\mathfrak{A}_n}) \subseteq A_{n:2}$, while $\Diamond^{\mathfrak{A}_n}{\restriction}A_{n:2}$ is diagonal. Thus, $\mathfrak{A}_n \in \mathsf{GKA}_n$. Conversely, consider any [finite] $\mathfrak{B} \in \mathsf{GKA}_n$ and any $\bar{a} \in (B^2 \setminus \Delta_B)$, in which case there is some $j \in 2$ such that $a_j \nleqslant^{\mathfrak{A}} a_{1-j}$, and so, by the Prime Ideal Theorem for distributive lattices, there is some prime filter $\mathcal{F}$ of $\mathfrak{B}$ such that $a \triangleq a_j \in \mathcal{F} \not\ni b \triangleq a_{1-j}$ (in particular, $\top^{\mathfrak{B}} \in \mathcal{F} \not\ni \bot^{\mathfrak{B}}$). Then, by (2.1), (2.2), (2.5), (3.21), (3.22) and (3.23), $g \triangleq \{\langle c, k\rangle \in (B \times n) \mid \tau_k^{\mathfrak{B}}(c) \in \mathcal{F}\} \in \hom(\mathfrak{B}, \mathfrak{A}_n)$. Let us prove, by contradiction, that $g(a) \neq g(b)$. For suppose $g(a) = g(b)$, in which case, for each $i \in n$, $(\tau_i^{\mathfrak{B}}(a) \in \mathcal{F}) \Leftrightarrow (\tau_i^{\mathfrak{B}}(b) \in \mathcal{F})$, and so, by (3.26), $(\tau_i^{\mathfrak{B}}(a) \equiv^{\mathfrak{B}} \tau_i^{\mathfrak{B}}(b)) \in \mathcal{F}$. Hence, by (3.24), $(a \leftrightarrow^{\mathfrak{B}} b) \in \mathcal{F}$. Therefore, by (3.25), $b \in \mathcal{F}$, for $a \in \mathcal{F}$. This contradiction shows that $g(a) \neq g(b)$. In this way, $H \triangleq \hom(\mathfrak{B}, \mathfrak{A}_n)$ is a [finite] set, while $e : B \to n^H, d \mapsto \langle h(d)\rangle_{h \in H}$ is an embedding of $\mathfrak{B}$ into $\mathfrak{A}_n^H$, as required. ⊣

Completing this discussion, we specify what is the set $\Im_{n,\mathcal{A}_n,i,+}$, where $i \in n$, in which case $\Im_{n,\mathcal{A}_n,i,-} = (\Im_n \setminus \Im_{n,\mathcal{A}_n,i,+})$ is equally specified. First, in case $i = ([n-1-](n-1))$, we clearly have $\Im_{n,\mathcal{A}_n,i,+} = (\{[\sim]x_0\} \cup \{\partial_j[\sim]x_0 \mid j \in N_{n-}\}))$. Now, assume $i \in ((n-1) \setminus 1)$, in which case $(n-1-i) \in ((n-1) \setminus 1)$, and so $\{i, n-1-i\} \subseteq D^{\mathcal{A}_n}$. Moreover, if $i \in N_{n-}$, then we have $i \leqslant m_n$, in which case, for all $j \in N_{n-}$, we get $j \leqslant m_n < (n-1-m_n) \leqslant (n-1-i)$, and so $\Im_{n,\mathcal{A}_n,i,+} = (\{x_0, \sim x_0\} \cup \{\partial_j x_0 \mid j \in N_{n-}, j < i\} \cup \{\partial_j \sim x_0 \mid j \in N_{n-}\})$. Likewise, if $(n-1-i) \in N_{n-}$, then $\Im_{n,\mathcal{A}_n,i,+} = (\{x_0, \sim x_0\} \cup \{\partial_j \sim x_0 \mid j \in N_{n-}, j < (n-1-i)\} \cup \{\partial_j x_0 \mid j \in N_{n-}\})$. Otherwise, we have $(2 \cdot i) = (n-1)$ is even (in particular, n is odd), in which case, for all $j \in N_{n-}$, we get $j < \frac{n-1}{2} = i = (n-1-i)$, and so $\Im_{n,\mathcal{A}_n,i,+} = \Im_n$. In this way, we have a transparent analytical expression for τ_i, making the above finite equational axiomatization of $\mathbf{QV}(\mathfrak{A}_n)$ equally transparent and algebraically natural. Nevertheless, we present below an alternative finite equational axiomatization of it that seems to better reflect its algebraic (as well as lattice-theoretic) substance and does not involve the secondary unary operators τ_i.

For every $j \in (((n-1) \setminus 1) \setminus N_n)$, we have $(n-1-j) \in N_n$, and so the secondary unary operation $\nabla_j x_0 \triangleq \sim\nabla_{n-1-j} x_0$.

THEOREM 3.19. $\mathbf{QV}(\mathfrak{A}_n)$ *is axiomatized by the identities axiomatizing the variety of Kleene algebras collectively with* (3.25) *and the following additional ones:*

$$(x_0 \leftrightarrow x_0) \approx \top, \tag{3.27}$$

$$((x_0 \leftrightarrow x_1) \wedge (x_1 \leftrightarrow x_2)) \lessapprox (x_0 \leftrightarrow x_2), \tag{3.28}$$

$$(x_0 \leftrightarrow x_1) \lessapprox (\wr x_0 \leftrightarrow \wr x_1) : \quad \wr \in (\Sigma_n \setminus \Sigma_{+,01}) \tag{3.29}$$

$$(x_0 \leftrightarrow x_1) \lessapprox ((x_0 \wedge x_2) \leftrightarrow (x_1 \wedge x_2)), \tag{3.30}$$

$$\top \approx ((x_0 \leftrightarrow (x_0 \wedge x_1)) \vee (x_1 \leftrightarrow (x_0 \wedge x_1))), \tag{3.31}$$

$$\sim((x_0 \leftrightarrow \bot) \vee (x_0 \leftrightarrow \top)) \approx (\vee\langle x_0 \leftrightarrow \nabla_{j''+1}(x_0)\rangle_{j'' \in (n-2)}), \tag{3.32}$$

$$\bot \approx ((x_0 \leftrightarrow x_1) \wedge \sim(x_0 \leftrightarrow x_1)), \tag{3.33}$$

$$(\nabla_{j'}(x_0) \leftrightarrow \nabla_j(x_0)) \lessapprox ((x_0 \leftrightarrow \bot) \vee (x_0 \leftrightarrow \top)) : \quad j' \neq j, \tag{3.34}$$

$$\nabla_\jmath \bot \approx \bot, \tag{3.35}$$

$$\nabla_\jmath \top \approx \top, \tag{3.36}$$

$$\partial_i \bot \approx \bot, \tag{3.37}$$

$$\partial_i \top \approx \top, \tag{3.38}$$

$$(\nabla_j x_0 \leftrightarrow \nabla_j \sim x_0) \approx (\sim(x_0 \leftrightarrow \bot) \wedge \sim(x_0 \leftrightarrow \top)), \tag{3.39}$$

$$(\sim\nabla_j x_0 \leftrightarrow \nabla_{n-1-j} x_0) \approx (\sim(x_0 \leftrightarrow \bot) \wedge \sim(x_0 \leftrightarrow \top)), \tag{3.40}$$

$$\nabla_\jmath \nabla_\jmath x_0 \approx \nabla_\jmath x_0, \tag{3.41}$$

$$(\partial_i \nabla_j x_0 \leftrightarrow \bot) \approx (\sim(x_0 \leftrightarrow \top)) : \quad j \leqslant i, \tag{3.42}$$

$$(\partial_i \nabla_j x_0 \leftrightarrow \top) \approx (\sim(x_0 \leftrightarrow \bot)) : \quad j \nleqslant i, \tag{3.43}$$

$$\begin{aligned}(x_2 \leftrightarrow x_3) \gtrapprox\ & (\wedge\langle((x_{2+k} \leftrightarrow \nabla_j x_k) \\ & \wedge(\sim(x_k \leftrightarrow \bot) \wedge \sim(x_k \leftrightarrow \top)))\rangle_{k \in 2},\end{aligned} \tag{3.44}$$

$$\begin{aligned}& (\nabla_{min(j,j')}(x_0 \wedge x_1) \leftrightarrow (\nabla_j x_0 \wedge \nabla_{j'} x_1)) \\ & \gtrapprox (\wedge\langle\sim(x_k \leftrightarrow \bot) \wedge \sim(x_k \leftrightarrow \top)\rangle_{k \in 2}),\end{aligned} \tag{3.45}$$

where $i \in N_{n-}$, $j, j' \in ((n-1) \setminus 1)$ *and* $\jmath \in N_n$.

Proof. The fact that $\mathfrak{A}_n$ satisfies the above identities is immediate, with using (3.12). Conversely, consider any Σ_n-algebra $\mathfrak{B}$ satisfying the identities involved and any $\bar{a} \in (B^2 \setminus \Delta_B)$, in which case there is some $k \in 2$ such that $a_k \nleqslant^{\mathfrak{A}} a_{1-k}$, and so, by the Prime Ideal Theorem for distributive lattices, there is some prime filter $\mathcal{F}$ of $\mathfrak{B}$ such that $a \triangleq a_k \in \mathcal{F} \not\ni b \triangleq a_{1-k}$ (in particular, $\top^{\mathfrak{B}} \in \mathcal{F} \not\ni \bot^{\mathfrak{B}}$). Then, by the commutativity identity for $\wedge$, (2.1), (2.2), (3.25), (3.27), (3.28), (3.29) and (3.30), $\{\langle \bot^{\mathfrak{B}}, \top^{\mathfrak{B}}\rangle, \langle a, b\rangle\}$ is disjoint with $\theta \triangleq \{\langle c, d\rangle \in B^2 \mid (c \leftrightarrow^{\mathfrak{B}} d) \in \mathcal{F}\} \in \mathrm{Con}(\mathfrak{B})$. Let $g \triangleq (\{\langle e, 0\rangle \mid B \ni e\ \theta\ \bot^{\mathfrak{B}}\} \cup \{\langle e, n-1\rangle \mid B \ni e\ \theta\ \top^{\mathfrak{B}}\} \cup \{\langle e, i\rangle \mid e \in B, i \in ((n-1) \setminus 1), e\ \theta\ \nabla_i^{\mathfrak{B}} e, \langle e, \bot^{\mathfrak{B}}\rangle \notin \theta \not\ni \langle e, \top^{\mathfrak{B}}\rangle\}$. Clearly, $(\mathrm{img}\, g) \subseteq n$, Moreover, by (2.1), (2.2), (2.3), (2.5), (3.32) and (3.33), $(\mathrm{dom}\, g) = B$.

Further, consider any $e \in B$. Then, as $\langle \bot^{\mathfrak{B}}, \top^{\mathfrak{B}}\rangle \notin \theta$, by (2.2), (3.32) and (3.33), $g[\{e\}]$ is a singleton, unless it is disjoint with $A_{n:2}$. Otherwise, consider any any $i, j \in ((n-1) \setminus 1)$ such that $e\ \theta\ \nabla_i^{\mathfrak{B}} e\ \theta\ \nabla_j^{\mathfrak{B}} e$. Then, by (3.34), $i = j$. Thus, $g : B \to n$. Moreover, by (3.27), $g((\bot|\top)^{\mathfrak{B}}) = (\bot|\top)^{\mathfrak{A}_n}$. Now, consider any $e \in B$, any $i \in N_{n-}$ and any $j \in N_n$. Then, in case $g(e) = (0|(n-1))$, that is, $e\ \theta\ (\bot|\top)^{\mathfrak{B}}$, by (3.35)|(3.36), we have $\nabla_j^{\mathfrak{B}} e\ \theta\ (\bot|\top)^{\mathfrak{B}}$, that is, $g(\nabla_j^{\mathfrak{B}} e) = (0|(n-1)) = \nabla_j^{\mathfrak{A}_n} g(e)$, while, by (2.5)|(2.6), we have $\sim^{\mathfrak{B}} e\ \theta\ (\top|\bot)^{\mathfrak{B}}$, that is, $g(\sim^{\mathfrak{B}} e) = ((n-1)|0) = \sim^{\mathfrak{A}_n} g(e)$, whereas, by (3.37)|(3.38), we have $\partial_i^{\mathfrak{B}} e\ \theta\ (\bot|\top)^{\mathfrak{B}}$, that is, $g(\partial_i^{\mathfrak{B}} e) = (0|(n-1)) = \partial_i^{\mathfrak{A}_n} g(e)$. Otherwise, by (3.41), we have $\nabla_j^{\mathfrak{B}} \nabla_j^{\mathfrak{B}} e\ \theta\ \nabla_j^{\mathfrak{B}} e$, and so we get $g(\nabla_j^{\mathfrak{B}} e) = j = \nabla_j^{\mathfrak{A}_n} g(e)$, while there is some $\jmath \in ((n-1) \setminus 1)$ such that $g(e) = \jmath$ and $e\ \theta\ \nabla_\jmath^{\mathfrak{B}} e$, in which case, by the commutativity identity for $\wedge$, (2.1), (2.3), (2.5), (3.33), (3.39) and (3.40), we have $\sim^{\mathfrak{B}} e\ \theta \sim^{\mathfrak{B}} \nabla_\jmath^{\mathfrak{B}} e\ \theta\ \nabla_{n-1-\jmath}^{\mathfrak{B}} e\ \theta\ \nabla_{n-1-\jmath}^{\mathfrak{B}} \sim^{\mathfrak{B}} e$, and so we get $g(\sim^{\mathfrak{B}} e) = (n-1-\jmath) = \sim^{\mathfrak{A}_n} g(e)$, whereas, in case $\jmath \leqslant | \nleqslant i$, by (2.1), (2.3), (2.5), (3.33) and (3.42)|(3.43), we have $\partial_i^{\mathfrak{B}} e\ \theta\ \partial_i^{\mathfrak{B}} \nabla_\jmath^{\mathfrak{B}} e\ \theta\ (\bot|\top)^{\mathfrak{B}}$, and so we get $g(\partial_i^{\mathfrak{B}} e) = (0|(n-1)) = \partial_i^{\mathfrak{A}_n} g(e)$.

Now, we prove that:

$$(3.46) \qquad (\ker g) = \theta.$$

For consider any $c, d \in B$. First, assume $c\ \theta\ d$. Then, in case $g(c) = (0|(n-1))$, we have $c\ \theta\ (\bot|\top)^{\mathfrak{B}}$, and so $d\ \theta\ (\bot|\top)^{\mathfrak{B}}$, that is, $g(d) = (0|(n-1)) = g(c)$, respectively. Otherwise, $\bot^{\mathfrak{B}}\ \not\theta\ c\ \not\theta\ \top^{\mathfrak{B}}$, while there is some $j \in ((n-1) \setminus 1)$ such that $g(c) = j$ and $c\ \theta\ \nabla_j^{\mathfrak{B}} c$, in which case $\bot^{\mathfrak{B}}\ \not\theta\ d\ \not\theta\ \top^{\mathfrak{B}}$, while $d\ \theta\ \nabla_j^{\mathfrak{B}} d$, and so $g(d) = j = g(c)$. Conversely, assume $g(c) = g(d)$. Then, in case $g(c) = g(d) = (0|(n-1))$, we have $c\ \theta\ (\bot|\top)^{\mathfrak{B}}\ \theta\ d$, respectively. Otherwise, $(c/d)\ \not\theta\ (\bot|\top)^{\mathfrak{B}}$, while there are some $i, j \in N_n$ such that $g(c|d) = (i|j)$ and $(c|d)\ \theta\ \nabla_{i|j}^{\mathfrak{B}}(c|d)$, respectively. In that case, since $g(c) = g(d)$, we have $i = j$. Then, by (2.3), (2.5), (3.33) and (3.44), we have $d\ \theta\ c$, and so (3.46) does hold.

Furthermore, we prove that g is monotonic with respect to the lattice partial ordering, that is:

$$(3.47) \qquad (c \leqslant^{\mathfrak{B}} d) \Rightarrow (g(c) \leqslant g(d)),$$

for all $c, d \in B$. For consider any $c, d \in B$ such that $c \leqslant^{\mathfrak{B}} d$ and the following complementary cases:

- $g(c) = 0$.
  Then, $g(c) = 0 \leqslant g(d)$.
- $g(c) \neq 0$,
  in which case $c\ \not\theta\ \bot^{\mathfrak{B}}$, and so $d\ \not\theta\ \bot^{\mathfrak{B}}$, that is, $g(d) \neq 0$, for, otherwise, it would hold that $c = (c \wedge^{\mathfrak{B}} d)\ \theta\ (c \wedge^{\mathfrak{B}} \bot^{\mathfrak{B}}) = \bot^{\mathfrak{B}}$. Consider the following complementary subcases:
  - $g(d) = (n-1)$.
    Then, $g(c) \leqslant (n-1) = g(d)$.
  - $g(d) \neq (n-1)$,
    in which case $d\ \not\theta\ \top^{\mathfrak{B}}$, and so $c\ \not\theta\ \top^{\mathfrak{B}}$, that is, $g(c) \neq (n-1)$, for,

otherwise, it would hold that $\top^{\mathfrak{B}} = (d \vee^{\mathfrak{B}} \top^{\mathfrak{B}})\ \theta\ (d \vee^{\mathfrak{B}} c) = d$. Then, there is some $(i|j) \in N_n$ such that $g(c|d) = (i|j)$ and $(c|d)\ \theta\ \nabla^{\mathfrak{B}}_{i|j}(c|d)$, respectively. Hence, by (2.1), (2.3), (2.5), (3.33) and (3.45), we have $\nabla^{\mathfrak{B}}_{\min(i,j)}(c) = \nabla^{\mathfrak{B}}_{\min(i,j)}(c \wedge^{\mathfrak{B}} d)\ \theta\ (\nabla^{\mathfrak{B}}_i c \wedge^{\mathfrak{B}} \nabla^{\mathfrak{B}}_j d)\ \theta\ (c \wedge^{\mathfrak{B}} d) = c\ \theta\ \nabla^{\mathfrak{B}}_i c$, and so, by (3.34), we get $i = \min(i,j)$, that is, $g(c) = i \leqslant j = g(d)$.

Thus, anyway, $g(c) \leqslant g(d)$, and so (3.47) holds.

Finally, consider any $c, d \in B$, in which case, by (3.31), we have $(c \wedge^{\mathfrak{B}} d)\ \theta\ (c|d)$, and so, by (3.46), we get $g(c \wedge^{\mathfrak{B}} d) = g(c|d)$. Moreover, $(c \wedge^{\mathfrak{B}} d) \leqslant^{\mathfrak{B}} (d|c)$, in which case, by (3.47), we have $g(c \wedge^{\mathfrak{B}} d) \leqslant g(d|c)$, and so, by the above equality, we get $g(c \wedge^{\mathfrak{B}} d)) = \min(g(c \wedge^{\mathfrak{B}} d), g(d|c)) = \min(g(c|d), g(d|c)) = \min(g(c), g(d)) = (g(c) \wedge^{\mathfrak{A}_n} g(d))$.

In this way, by (2.1) and (2.2), $g \in \hom(\mathfrak{B}, \mathfrak{A}_n)$. Moreover, by (3.46), $g(a) \neq g(b)$. Thus, $H \triangleq \hom(\mathfrak{B}, \mathfrak{A}_n)$ is a set, while $f : B \to n^H, e \mapsto \langle h(e) \rangle_{h \in H}$ is an embedding of $\mathfrak{B}$ into $\mathfrak{A}_n^H$, and so $\mathfrak{B} \in \mathbf{QV}(\mathfrak{A}_n)$, as required. ⊣

However, this finite equational axiomatization of $\mathbf{QV}(\mathfrak{A}_n)$ is not most intrinsic because of involving secondary unary operators ∇_j, where $j \in (((n-1) \setminus 1) \setminus N_n)$. Below, we find that without involving these.

Theorem 3.20. *$\mathbf{QV}(\mathfrak{A}_n)$ is axiomatized by the identities axiomatizing the variety of Kleene algebras collectively with* (3.25)*,* (3.27)*,* (3.28)*,* (3.29)*,* (3.30)*,* (3.31)*,* (3.33)*,* (3.35)*,* (3.36)*,* (3.37)*,* (3.38) *as well as both* (3.34)*,* (3.39)*,* (3.42)*,* (3.43)*,* (3.44)*,* (3.45) *but with $j, j' \in N_n$ and* (3.40) *but with $(n-1-j) = \jmath \in N_n$ and the following additional identities:*

$$(\sim(x_0 \leftrightarrow \bot) \wedge \sim(x_0 \leftrightarrow \top)) \approx (\vee \langle (x_0 \wedge \sim x_0) \leftrightarrow \nabla_{j'+1}(x_0) \rangle_{j' \in l_n}), \tag{3.48}$$

$$((x_0 \leftrightarrow \sim x_0) \approx \sim(x_0 \leftrightarrow \nabla_j x_0) : \quad (n-1) \neq (2 \cdot j), \tag{3.49}$$

$$\sim(x_0 \leftrightarrow \top) \approx ((\nabla_\jmath x_0 \wedge \sim \nabla_\jmath x_0) \leftrightarrow \nabla_\jmath x_0), \tag{3.50}$$

$$\nabla_j \nabla_\jmath x_0 \approx \nabla_\jmath x_0, \tag{3.51}$$

$$\partial_i(x_0 \vee \sim x_0) \approx \top, \tag{3.52}$$

$$(\nabla_{max(j,\jmath)}(x_0 \vee x_1) \leftrightarrow (\nabla_j x_0 \vee \nabla_\jmath x_1)) \gtrapprox (\wedge \langle \sim(x_k \leftrightarrow \bot) \wedge \sim(x_k \leftrightarrow \top) \rangle_{k \in 2}), \tag{3.53}$$

where $i \in N_{n-}$ and $j, \jmath \in N_n$.

Proof. The fact that $\mathfrak{A}_n$ satisfies the above identities is immediate, with using (3.12). Conversely, consider any Σ_n-algebra $\mathfrak{B}$ satisfying the identities involved and any $\bar{a} \in (B^2 \setminus \Delta_B)$, in which case there is some $k \in 2$ such that $a_k \nleqslant^{\mathfrak{A}} a_{1-k}$, and so, by the Prime Ideal Theorem for distributive lattices, there is some prime filter $\mathcal{F}$ of $\mathfrak{B}$ such that $a \triangleq a_k \in \mathcal{F} \not\ni b \triangleq a_{1-k}$ (in particular, $\top^{\mathfrak{B}} \in \mathcal{F} \not\ni \bot^{\mathfrak{B}}$). Then, by the commutativity identity for

$\wedge$, (2.1), (2.2), (3.25), (3.27), (3.28), (3.29) and (3.30), $\{\langle\bot^{\mathfrak{B}},\top^{\mathfrak{B}}\rangle,\langle a,b\rangle\}$ is disjoint with $\theta\triangleq\{\langle c,d\rangle\in B^2\mid(c\leftrightarrow^{\mathfrak{B}}d)\in\mathcal{F}\}\in\mathrm{Con}(\mathfrak{B})$. Let $g\triangleq(\{\langle e,0\rangle\mid B\ni e\;\theta\;\bot^{\mathfrak{B}}\}\cup\{\langle e,n-1\rangle\mid B\ni e\;\theta\;\top^{\mathfrak{B}}\}\cup\{\langle e,(n-1-)^m i\rangle\mid e\in B,m\in 2,i\in N_n,(\sim^{\mathfrak{B}})^m e\;\theta\;(e\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}e)\;\theta\;\nabla_i^{\mathfrak{B}}e,\langle e,\bot^{\mathfrak{B}}\rangle\notin\theta\not\ni\langle e,\top^{\mathfrak{B}}\rangle\}$. Clearly, $(\mathrm{img}\,g)\subseteq n$, Moreover, by (2.1), (2.2), (2.3), (2.5), (3.31), (3.33) and (3.48), $(\mathrm{dom}\,g)=B$. Next, consider any $e\in B$. Then, as $\langle\bot^{\mathfrak{B}},\top^{\mathfrak{B}}\rangle\notin\theta$, by (3.33) and (3.48), $g[\{e\}]$ is a singleton, unless it is disjoint with $A_{n:2}$. Otherwise, consider any $l,m\in 2$ and any $i,j\in N_n$ such that $(\sim^{\mathfrak{B}})^l e\;\theta\;(\sim^{\mathfrak{B}})^m e\;\theta\;(e\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}e)\;\theta\;\nabla_i^{\mathfrak{B}}e\;\theta\;\nabla_j^{\mathfrak{B}}e$. Then, by (3.34), $i=j$. Therefore, $(n-1-)^l i=(n-1-)^m j$, whenever $m=l$. Otherwise, $\sim^{\mathfrak{B}}e\;\theta\;e$, in which case, by (3.33) and (3.49), $(n-1)=(2\cdot i)=(2\cdot j)$, and so $(n-1-)^l i=i=j=(n-1-)^m j$. Thus, $g:B\to n$. Moreover, by (3.27), $g((\bot|\top)^{\mathfrak{B}})=(0|(n-1))=(\bot|\top)^{\mathfrak{A}_n}$.

Further, consider any $e\in B$, any $i\in N_{n-}$ and any $j\in N_n$. Then, in case $g(e)=(0|(n-1))$, that is, $e\;\theta\;(\bot|\top)^{\mathfrak{B}}$, by (3.35)|(3.36), we have $\nabla_j^{\mathfrak{B}}e\;\theta\;(\bot|\top)^{\mathfrak{B}}$, and so $g(\nabla_j^{\mathfrak{B}}e)=(0|(n-1))=\nabla_j^{\mathfrak{A}_n}g(e)$, while, by (2.5)|(2.6), we have $\sim^{\mathfrak{B}}e\;\theta\;(\top|\bot)^{\mathfrak{B}}$, and so $g(\sim^{\mathfrak{B}}e)=((n-1)|0)=\sim^{\mathfrak{A}_n}g(e)$, whereas, by (3.37)|(3.38), we have $\partial_i^{\mathfrak{B}}e\;\theta\;(\bot|\top)^{\mathfrak{B}}$, and so $g(\partial_i^{\mathfrak{B}}e)=(0|(n-1))=\partial_i^{\mathfrak{A}_n}g(e)$. Otherwise, by (2.1), (2.3), (2.5), (3.33), (3.50) and (3.51), we have $\nabla_j^{\mathfrak{B}}\nabla_j^{\mathfrak{B}}e\;\theta\;\nabla_j^{\mathfrak{B}}e\;\theta\;(\nabla_j^{\mathfrak{B}}e\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}\nabla_j^{\mathfrak{B}}e)$, and so we get $g(\nabla_j^{\mathfrak{B}}e)=j=\nabla_j^{\mathfrak{A}_n}g(e)$, while there are some $m\in 2$ and some $\jmath\in N_n$ such that $g(e)=(n-1-)^m\jmath$ and $(\sim^{\mathfrak{B}})^m e\;\theta\;(e\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}e)\;\theta\;\nabla_\jmath^{\mathfrak{B}}e$, in which case, by the commutativity identity for $\wedge$, (2.1), (2.3), (2.5), (3.33) and (3.39), we have $(\sim^{\mathfrak{B}})^{1-m}\sim^{\mathfrak{B}}e\;\theta\;(\sim^{\mathfrak{B}}e\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}\sim^{\mathfrak{B}}e)\;\theta\;\nabla_\jmath^{\mathfrak{B}}\sim^{\mathfrak{B}}e$, and so we get $g(\sim^{\mathfrak{B}}e)=(n-1-)^{1-m}\jmath=\sim^{\mathfrak{A}_n}g(e)$, whereas, in case $m=1$, by the commutativity identity for $\wedge$, (2.1), (2.3) and (3.52), we have $\partial_i^{\mathfrak{B}}e\;\theta\;\partial_i^{\mathfrak{B}}(e\vee^{\mathfrak{B}}\sim^{\mathfrak{B}}e)=\top^{\mathfrak{B}}$, and so we get $g(\partial_i^{\mathfrak{B}}e)=(n-1)=\partial_i^{\mathfrak{A}_n}g(e)$, and, in case $m=0$ and $\jmath\leqslant|\nleqslant i$, by (2.1), (2.3), (2.5), (3.33) and (3.42)|(3.43), we have $\partial_i^{\mathfrak{B}}e\;\theta\;\partial_i^{\mathfrak{B}}\nabla_\jmath^{\mathfrak{B}}e\;\theta\;(\bot|\top)^{\mathfrak{B}}$, and so we get $g(\partial_i^{\mathfrak{B}}e)=(0|(n-1))=\partial_i^{\mathfrak{A}_n}g(e)$.

Furthermore, we prove that:

$$(\ker g)=\theta. \tag{3.54}$$

For consider any $c,d\in B$. First, assume $c\;\theta\;d$. Then, in case $g(c)=(0|(n-1))$, we have $c\;\theta\;(\bot|\top)^{\mathfrak{B}}$, and so $d\;\theta\;(\bot|\top)^{\mathfrak{B}}$, that is, $g(d)=(0|(n-1))=g(c)$, respectively. Otherwise, $\bot^{\mathfrak{B}}\not\theta\;c\not\theta\;\top^{\mathfrak{B}}$, while there are some $m\in 2$ and some $j\in N_n$ such that $g(c)=(n-1-)^m j$ and $(\sim^{\mathfrak{B}})^m c\;\theta\;(c\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}c)\;\theta\;\nabla_j^{\mathfrak{B}}c$, in which case $\bot^{\mathfrak{B}}\not\theta\;d\not\theta\;\top^{\mathfrak{B}}$, while $(\sim^{\mathfrak{B}})^m d\;\theta\;(d\wedge^{\mathfrak{B}}\sim^{\mathfrak{B}}d)\;\theta\;\nabla_j^{\mathfrak{B}}d$, and so $g(d)=(n-1-)^m j=g(c)$. Conversely, assume $g(c)=g(d)$. Then, in case $g(c)=g(d)=(0|(n-1))$, we have $c\;\theta\;(\bot|\top)^{\mathfrak{B}}\;\theta\;d$, respectively. Otherwise, $(c/d)\not\theta\;(\bot|\top)^{\mathfrak{B}}$, while there are some $l, m \in 2$ and some $i, j \in N_n$ such that $g(c|d) = (n-1-)^{l|m}(i|j)$ and $(\sim^{\mathfrak{B}})^{l|m}(c|d)\ \theta\ \nabla^{\mathfrak{B}}_{i|j}(c|d)$, respectively. In that case, since $(n-1-)^l i = (n-1-)^m j$, we have $i = j$, if $l = m$, while, otherwise, as $(i|j) \in N_n$, we have $(i|j) \leqslant ((n-1-i)|(n-1-j)) = (j|i)$, and so get $i = j = (n-1-i)$. Then, in case $\max(l-m, m-l) = (0[+1])$, by (2.1), (2.3), (2.5), (3.33) and (3.44) [as well as (3.40)], we have $d\ \theta\ [\sim^{\mathfrak{B}}]c[\theta\ c]$ [for $(\sim^{\mathfrak{B}})^{1-l}c = \sim^{\mathfrak{B}}(\sim^{\mathfrak{B}})^l c\ \theta \sim^{\mathfrak{B}}\nabla^{\mathfrak{B}}_i c\ \theta\ \nabla^{\mathfrak{B}}_i c\ \theta\ (\sim^{\mathfrak{B}})^l c$], and so (3.54) does hold.

Now, we are in a position to prove that g is monotonic with respect to the lattice partial ordering, that is:

$$(c \leqslant^{\mathfrak{B}} d) \Rightarrow (g(c) \leqslant g(d)), \tag{3.55}$$

for all $c, d \in B$. For consider any $c, d \in B$ such that $c \leqslant^{\mathfrak{B}} d$ and the following complementary cases:

- $g(c) = 0$.
  Then, $g(c) = 0 \leqslant g(d)$.
- $g(c) \neq 0$,
  in which case $c \not\theta \perp^{\mathfrak{B}}$, and so $d \not\theta \perp^{\mathfrak{B}}$, that is, $g(d) \neq 0$, for, otherwise, it would hold that $c = (c \wedge^{\mathfrak{B}} d)\ \theta\ (c \wedge^{\mathfrak{B}} \perp^{\mathfrak{B}}) = \perp^{\mathfrak{B}}$. Consider the following complementary subcases:
  - $g(d) = (n-1)$.
    Then, $g(c) \leqslant (n-1) = g(d)$.
  - $g(d) \neq (n-1)$,
    in which case $d \not\theta \top^{\mathfrak{B}}$, and so $c \not\theta \top^{\mathfrak{B}}$, that is, $g(c) \neq (n-1)$, for, otherwise, it would hold that $\top^{\mathfrak{B}} = (d \vee^{\mathfrak{B}} \top^{\mathfrak{B}})\ \theta\ (d \vee^{\mathfrak{B}} c) = d$. Then, there are some $(l|m) \in 2$ and some $(i|j) \in N_n$ such that $g(c|d) = (n-1-)^{l|m}(i|j)$ and $(\sim^{\mathfrak{B}})^{l|m}(c|d)\ \theta\ (((c|d) \wedge^{\mathfrak{B}} \sim^{\mathfrak{A}}(c|d))\ \theta\ \nabla^{\mathfrak{B}}_{i|j}(c|d)$, respectively. Consider the following complementary subsubcases:
    * $l = 0$.
      Consider the following complementary subsubsubcases:
      · $m = 1$.
      Then, as $i, j \in N_n \ni l_n$, and so $i \leqslant l_n \geqslant j$, we have $g(c) = i \leqslant l_n \leqslant (n-1-l_n) \leqslant (n-1-j) = g(d)$.
      · $m = 0$.
      Then, by (2.1), (2.3), (2.5), (3.33) and (3.45), we have $\nabla^{\mathfrak{B}}_{\min(i,j)}(c) = \nabla^{\mathfrak{B}}_{\min(i,j)}(c \wedge^{\mathfrak{B}} d)\ \theta\ (\nabla^{\mathfrak{B}}_i c \wedge^{\mathfrak{B}} \nabla^{\mathfrak{B}}_j d)\ \theta\ (c \wedge^{\mathfrak{B}} d) = c\ \theta\ \nabla^{\mathfrak{B}}_i c$, and so, by (3.34), we get $i = \min(i, j)$, that is, $g(c) = i \leqslant j = g(d)$.
    * $l = 1$.
      Consider the following complementary subsubsubcases:
      · $m = 0$.
      Then, by the commutativity identity for $\wedge$, (2.1) and

(2.2), we have $c\ \theta\ (c \vee \sim^{\mathfrak{A}} c)$, in which case, by (2.4), we get $d\ \theta\ (d \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} d) = ((d \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} d) \wedge^{\mathfrak{B}} (c \vee \sim^{\mathfrak{A}} c))\ \theta\ (d \wedge^{\mathfrak{B}} c) = c$, and so, by (3.54), we eventually get $g(c) = g(d) \leqslant g(d)$.

· $m = 1$.
Then, by (2.1), (2.2), (2.3), (2.5), (3.33) and (3.45)/(3.53), we have $\nabla^{\mathfrak{B}}_{(\min/\max)(i,j)}(c/d) = \nabla^{\mathfrak{B}}_{(\min/\max)(i,j)}(c(\wedge/\vee)^{\mathfrak{B}} d)\ \theta\ (\nabla^{\mathfrak{B}}_i c(\wedge/\vee)^{\mathfrak{B}} \nabla^{\mathfrak{B}}_j d)\ \theta\ (\sim^{\mathfrak{B}} c(\wedge/\vee)^{\mathfrak{B}} \sim^{\mathfrak{B}} d) = \sim^{\mathfrak{B}}(c(\vee/\wedge)^{\mathfrak{B}} d) = \sim^{\mathfrak{B}}(d/c)\ \theta\ \nabla^{\mathfrak{B}}_{j/i}(d/c)$, respectively, in which case, applying $\nabla^{\mathfrak{B}}_i$ to the first equivalence, by (3.51), we get $\nabla^{\mathfrak{B}}_i c\ \theta\ \nabla^{\mathfrak{B}}_i d$, and so, combining this with the second equivalence, by (3.34), we eventually get $i = \max(i,j)$, that is, $g(c) = (n-1-i) \leqslant (n-1-j) = g(d)$.

Thus, anyway, $g(c) \leqslant g(d)$, and so (3.55) holds.

Finally, consider any $c, d \in B$, in which case, by (3.31), we have $(c \wedge^{\mathfrak{B}} d)\ \theta\ (c|d)$, and so, by (3.54), we get $g(c \wedge^{\mathfrak{B}} d) = g(c|d)$. Moreover, $(c \wedge^{\mathfrak{B}} d) \leqslant^{\mathfrak{B}} (d|c)$, in which case, by (3.55), we have $g(c \wedge^{\mathfrak{B}} d) \leqslant g(d|c)$, and so, by the above equality, we get $g(c \wedge^{\mathfrak{B}} d)) = \min(g(c \wedge^{\mathfrak{B}} d), g(d|c)) = \min(g(c|d), g(d|c)) = \min(g(c), g(d)) = (g(c) \wedge^{\mathfrak{A}_n} g(d))$.

In this way, by (2.1) and (2.2), $g \in \hom(\mathfrak{B}, \mathfrak{A}_n)$. Moreover, by (3.54), $g(a) \neq g(b)$. Thus, $H \triangleq \hom(\mathfrak{B}, \mathfrak{A}_n)$ is a set, while $f : B \to n^H, e \mapsto \langle h(e) \rangle_{h \in H}$ is an embedding of $\mathfrak{B}$ into $\mathfrak{A}_n^H$, and so $\mathfrak{B} \in \mathbf{QV}(\mathfrak{A}_n)$, as required. ⊣

Though the above finite equational axiomatizations essentially rely upon either tabular — (3.23) – or congruence — (3.27), (3.28), (3.29) and (3.30) — identities for the secondary equivalence connective $\leftrightarrow$, below we find that not involving these. Just to unify and abbreviate further notations, we use the secondary unary connectives $\partial_0 x_0 \triangleq \Diamond x_0$ satisfying (3.1) with $i = 0$, in view of (3.9), as well as, for each $\diamond \in \Sigma_+$, $\eth_\diamond x_0 \triangleq (\diamond\langle \sim\partial_i x_0 \diamond \partial_i \sim x_0 \rangle_{i \in (m_n+1)})$, in which case, by Lemma 3.1, $\Im'_n \triangleq \{\partial_i \sim^k x_0 \mid i \in (m_n+1), k \in 2\}$ is an equality determinant for $\mathcal{A}_n$ with $\Im'_{n,\mathcal{A}_n,0/(n-1),-|+} = \{\partial_i \sim^k x_0 \mid i \in (m_n+1)\}$, where $k = ((0|1)/(1|0))$, and so, by the $\wedge$-conjunctivity/$\vee$-disjunctivity of $\mathcal{A}_n$, we have:

$$\eth^{\mathfrak{A}_n}_{\wedge/\vee} m = \begin{cases} (n-1)/0 & \text{if } m = (0/(n-1)), \\ 0/(n-1) & \text{otherwise,} \end{cases} \tag{3.56}$$

for all $m \in n$.

Theorem 3.21. $\mathbf{QV}(\mathfrak{A}_n)$ *is axiomatized by the identities axiomatizing the variety of Kleene algebras collectively with* (3.25) *as well as both* (3.37), (3.38) *and* (3.52) *but with* $i \in (m_n+1)$ *and the following additional* *identities:*

$$(\partial_i \nabla_j x_0 \wedge \eth_\vee x_0) \approx \bot : \quad j \leqslant i, \tag{3.57}$$

$$(\partial_i \nabla_j x_0 \vee \eth_\wedge x_0) \approx \top : \quad i < j, \tag{3.58}$$

$$\sim\eth_\vee x_0 \lessapprox \partial_i \nabla_j x_0, \tag{3.59}$$

$$\partial_i \sim\nabla_j x_0 \lessapprox \eth_\vee x_0, \tag{3.60}$$

$$\eth_\wedge x_0 \lessapprox \partial_i \sim\nabla_j x_0, \tag{3.61}$$

$$\partial_i \nabla_j x_0 \lessapprox \sim\eth_\wedge x_0, \tag{3.62}$$

$$\partial_i \sim\nabla_j x_0 \gtrapprox \eth_\vee x_0, \tag{3.63}$$

$$\partial_\imath x_0 \lessapprox \partial_i x_0 : \quad i < \imath, \tag{3.64}$$

$$\partial_i (x_0 \wedge x_1) \approx (\partial_i x_0 \wedge \partial_i x_1), \tag{3.65}$$

$$\partial_i (x_0 \vee x_1) \approx (\partial_i x_0 \vee \partial_i x_1), \tag{3.66}$$

$$\partial_i \sim^k \partial_\imath x_0 \approx \sim^k \partial_\imath x_0, \tag{3.67}$$

$$(\partial_i x_0 \wedge \sim\partial_i x_0) \approx \bot, \tag{3.68}$$

$$(\partial_{m_n} x_0 \wedge \partial_{m_n} \sim x_0) \approx \bot : \quad (2 \cdot (m_n + 1)) \neq (n - 1), \tag{3.69}$$

where $k \in 2$, $i \in (m_n + 1)$, $\imath \in N_{n-}$ *and* $j \in N_n$.

Proof. The fact that $\mathfrak{A}_n$ satisfies the above identities but the seventh and last ones [including the first six ones] is immediate, using (3.1) with $i \in (m_n + 1)$ [and (3.56)], (3.63) being due to both (3.56) and the fact that $i < (n - 1 - j)$, where $i \in (m_n + 1)$ and $j \in N_n$, for, otherwise, since $(j|i) \leqslant (l|m)_n$ and $m_n \in N_{n-} \subseteq N_n \ni l_n$, we would have $l_n \leqslant (n - 1 - l_n) \leqslant (n - 1 - j) \leqslant i \leqslant m_n \leqslant l_n$, in which case we would get $m_n = l_n = (n-1-l_n)$, and so would eventually get $m_n < (n-1-m_n) = m_n$.

First, we prove (3.69) by contradiction. For suppose $(2 \cdot (m_n + 1)) \neq (n - 1)$ but (3.69) is not true in $\mathfrak{A}_n$ under some $[x_0/\imath]$, where $\imath \in n$, in which case both $m_n < \imath$ and $m_n < (n - 1 - \imath)$, and so both $(m_n + 1) \leqslant \imath$ and $(m_n + 1) \leqslant (n - 1 - \imath)$. Then, $(m_n + 1) \leqslant \jmath \triangleq \min(\imath, n - 1 - \imath) \leqslant \max(\imath, n - 1 - \imath) = (n - 1 - \jmath) \leqslant (n - 2 - m_n)$. Therefore, if $m_n + 1$ was not equal to $n - 2 - m_n$, then it would belong to N_{n-}, in which case it would be lesser or equal to m_n, and so 1 would be lesser or equal to 0. Hence, $(m_n + 1) = (n - 2 - m_n)$, in which case $(2 \cdot (m_n + 1)) = (n - 1)$, and so this contradiction shows that (3.69) is true in $\mathfrak{A}_n$.

Conversely, consider any Σ_n-algebra $\mathfrak{B}$ satisfying the identities involved and any $\bar{a} \in (B^2 \setminus \Delta_B)$, in which case $b \triangleq (a_0 \vee^{\mathfrak{B}} a_1) \nleqslant^{\mathfrak{B}} c \triangleq (a_0 \wedge^{\mathfrak{B}} a_1)$, and so, by the Prime Ideal Theorem for distributive lattices, there is some prime filter $\mathcal{F}$ of $\mathfrak{B}$ such that $b \in \mathcal{F} \not\ni c$ (in particular, $\varnothing \neq \mathcal{F} \neq B$, in which case $\top^{\mathfrak{B}} \in \mathcal{F} \not\ni \bot^{\mathfrak{B}}$, and so, for every Boolean $d \in B$, $(d \in \mathcal{F}) \Leftrightarrow (\sim^{\mathfrak{B}} d \notin \mathcal{F})$). Let $R \triangleq \{\langle d, \langle k, l \rangle\rangle \in (B \times (2 \times (l_n + 1))) \mid (\partial^{\mathfrak{B}}_{m_n} (\sim^{\mathfrak{B}})^{1-k} d \in \mathcal{F}, \forall m \in (m_n + 1) : (\partial^{\mathfrak{B}}_m (\sim^{\mathfrak{B}})^k d \in \mathcal{F}) \Leftrightarrow (m < l)\}$

and $g \triangleq \{\langle d, (n-1-)^k l\rangle \mid \langle d, \langle k, l\rangle\rangle \in R\}$. Clearly, $(\mathrm{img}\, g) \subseteq n$ and $(\mathrm{dom}\, g) \subseteq B \supseteq (\mathrm{dom}\, R)$. Conversely, consider any $d \in B$. Given any $k \in 2$, set $M_k(d) \triangleq \{m \in (m_n{+}1) \mid \partial^{\mathfrak{B}}_m (\sim^{\mathfrak{B}})^k d \notin \mathcal{F}\}$, then putting $M(d) \triangleq (M_0(d) \cup M_1(d))$, in which case, by (3.52) and (3.66), $(M_0(d) \cap M_1(d)) = \varnothing$, while, by (3.64), $(m_n \in M_k(d)) \Leftrightarrow (M_k(d) \neq \varnothing)$, and so either $M_0(d)$ or $M_1(d)$ is empty. Consider the following complementary cases:

- $M(d) \neq \varnothing$,
  in which case there is a unique $k \in 2$ such that $M_k(d) \neq \varnothing = M_{1-k}(d)$, and so $l \triangleq \min(M_k(d)) \in M_k(d) \subseteq (m_n + 1) \subseteq (l_n + 1)$, for $m_n \in N_{n-} \subseteq N_n$. Then, for any $m \in (m_n + 1)$, $m \geqslant l$, whenever $m \in M_k(d)$, while, as $l \in M_k(d)$, by (3.64), $m < l$, otherwise. In this way, since $M_{1-k}(d) = \varnothing$, $\langle d, \langle k, l\rangle\rangle \in R$.
- $M(d) = \varnothing$,
  in which case $m_n \notin M(d)$, and so, by (3.69), $(2 \cdot (m_n + 1)) = (n - 1)$. Then, n is odd, in which case $n - 1$ is even, and so $l \triangleq \frac{n-1}{2} = l_n$,[2] while $m_n < l$, and so $m < l$, for all $m \in (m_n + 1)$, whereas $k \triangleq 0 \in 2$. In this way, $\langle d, \langle k, l\rangle\rangle \in R$.

Thus, anyway, $\langle d, \langle \Bbbk(d), \ell(d)\rangle\rangle \in R$, where $\Bbbk(d) \triangleq (1 - \max\{k \in 2 \mid M_k(d) = \varnothing\})$ and

$$\ell(d) \triangleq \begin{cases} \min M_{\Bbbk(d)} & \text{if } M(d) \neq \varnothing, \\ \frac{n-1}{2} & \text{otherwise,} \end{cases}$$

in which case $\langle d, (n-1-)^{\Bbbk(d)} \ell(d)\rangle \in g$, and so $d \in ((\mathrm{dom}\, g) \cap (\mathrm{dom}\, R))$ (in particular, $(\mathrm{dom}\, g) = B = (\mathrm{dom}\, R) = \mathrm{dom}(R \circ \pi_1))$. Now, consider any $\bar{k} \in 2^2$ and any $\bar{l} \in (l_n{+}1)^2$ such that, for every $\jmath \in 2$, $\langle d, \langle k_\jmath, l_\jmath\rangle\rangle \in R$, and the following complementary cases:

- $k_0 = k_1$.
  Let us prove, by contradiction, that $l_0 = l_1$. For suppose $l_0 \neq l_1$, in which case $l_\jmath < l_{1-\jmath}$, for some $\jmath \in 2$, and so $l_\jmath \leqslant m \triangleq (l_{1-\jmath} {-} 1) < l_{1-\jmath}$. And what is more, $(2 \cdot m) = (2 \cdot (l_{1-\jmath} - 1)) = ((2 \cdot l_{1-\jmath}) - 2) \leqslant ((n-1) - 2) = (n - 3) < (n - 1)$, in which case $m \in (m_n + 1)$, and so $\partial^{\mathfrak{B}}_m (\sim^{\mathfrak{B}})^{k_\jmath} d \notin \mathcal{F} \ni \partial^{\mathfrak{B}}_m (\sim^{\mathfrak{B}})^{k_{1-\jmath}} d$. This contradicts to the fact that $k_0 = k_1$. Thus, $l_0 = l_1$, and so $(n-1-)^{k_0} l_0 = (n-1-)^{k_1} l_1$.
- $k_0 \neq k_1$,
  in which case $\{k_0, k_1\} = 2$, and so $\{1 - k_0, 1 - k_1\} = 2$. Then, $M_\jmath(d) = \varnothing \not\ni m_n$, for all $\jmath \in 2$, in which case, by (3.69), $(m_n + 1) = \frac{n-1}{2} = l_n$, and so $m_n < l_\jmath \leqslant l_n$, that is, $l_0 = l_n = l_1$ (in particular, $(n-1-)^{k_0} l_0 = l_n = (n-1-)^{k_1} l_1)$.

[2] From now on, the fact that $\frac{n-1}{2} \in n$ is supposed to subsume *tacitly/implicitly* the fact that n is odd.

In this way, $g : B \to n$, in which case $g(d) = (n-1-)^{\Bbbk(d)}\ell(d)$, while $(R \circ \pi_1) : B \to (l_n+1)$, in which case $(R \circ \pi_1)(d) = \ell(d)$. Moreover, by (2.5), (2.6), (3.37) and (3.38), $M_{0/1}(\bot^{\mathfrak{B}}|\top^{\mathfrak{B}}) = (((m_n+1)|\varnothing)/(\varnothing|(m_n+1)))$, in which case $\Bbbk(\bot^{\mathfrak{B}}|\top^{\mathfrak{B}}) = (0|1)$, while $\ell(\bot^{\mathfrak{B}}|\top^{\mathfrak{B}}) = 0$, and so $g(\bot^{\mathfrak{B}}|\top^{\mathfrak{B}}) = (0|(n-1)) = (\bot^{\mathfrak{A}_n}|\top^{\mathfrak{A}_n})$. Likewise, by (2.1), $M_{0/1}(\sim^{\mathfrak{B}}d) = M_{1/0}(d)$, in which case $\ell(\sim^{\mathfrak{B}}d) = \ell(d)$, while $\Bbbk(\sim^{\mathfrak{B}}d) = 0 = \Bbbk(d)$, if $M(\sim^{\mathfrak{B}}d) = M(d) = \varnothing$, whereas $\Bbbk(\sim^{\mathfrak{B}}d) = (1-\Bbbk(d))$, otherwise, and so $g(\sim^{\mathfrak{B}}d) = \sim^{\mathfrak{A}_n}g(d)$, for $(n-1-g(d)) = g(d)$, whenever $g(d) = \frac{n-1}{2}$.

Next, consider any $i \in N_{n-}$. Then, by (3.67) and (3.68), we have

$$(m \in M_{0[+1]}(\partial_i^{\mathfrak{B}}d) \Leftrightarrow (i \in ([(m_n+1)\backslash]M_0(d)), \tag{3.70}$$

for all $m \in (m_n+1)$. Consider the following complementary cases:

- $M_0(d) = \varnothing$.
  Then, by (3.70), we have $M_{0/1}(\partial_i^{\mathfrak{B}}d) = (\varnothing/(m_n+1))$, in which case we get $\Bbbk(\partial_i^{\mathfrak{B}}d) = 1$, while $\ell(\partial_i^{\mathfrak{B}}d) = \min(m_n+1) = 0$, and so $g(\partial_i^{\mathfrak{B}}c) = (n-1)$. Consider the following complementary subcases:
  - $M_1(d) = \varnothing$.
    Then, $M(d) = \varnothing$, in which case $g(d) = \ell(d) = \frac{n-1}{2}$, and so $i \leqslant m_n < g(d)$, for $m_n \in N_{n-}$.
  - $M_1(d) \neq \varnothing$.
    Then, $\Bbbk(d) = 1$, $\ell(d) \in (l_n+1)$ and $g(d) = (n-1-\ell(d))$. Consider the following complementary subsubcases:
    * $m_n = l_n$,
      in which case $\ell(d) \leqslant m_n$, and so $i \leqslant m_n < (n-1-m_n) \leqslant g(d)$, for $m_n \in N_{n-}$.
    * $m_n \neq l_n$,
      in which case $m_n < l_n$, for $m_n \leqslant l_n$, because $N_{n-} \subseteq N_n$, and so, as $\ell(d) \leqslant l_n \in N_n$, $i \leqslant m_n < l_n \leqslant (n-1-l_n) \leqslant g(d)$.

  Thus, anyway, $i < g(d)$, and so $\partial_i^{\mathfrak{A}_n}g(d) = (n-1) = g(\partial_i^{\mathfrak{B}}d)$.
- $M_0(d) \neq \varnothing$.
  Then, $\Bbbk(d) = 0$, in which case $g(d) = \ell(d)$, and so, by (3.1), we have $(i \in | \notin M_0(d)) \Leftrightarrow (g(d) \leqslant | > i) \Leftrightarrow (\partial_i^{\mathfrak{B}}g(d) = (0|(n-1)))$. Hence, by (3.70), providing $\partial_i^{\mathfrak{A}_n}g(d) = (0|(n-1))$, we get $M_{0/1}(\partial_i^{\mathfrak{B}}d) = (((m_n+1)|\varnothing)/(\varnothing|(m_n+1)))$, in which case $\Bbbk(\partial_i^{\mathfrak{B}}d) = (0|1)$, while $\ell(\partial_i^{\mathfrak{B}}d) = \min(m_n+1) = 0$, and so $g(\partial_i^{\mathfrak{B}}d) = (0|(n-1)) = \partial_i^{\mathfrak{A}_n}g(d)$.

Further, consider any $j \in N_n$ and the following complementary cases:

- both of $\eth_{\vee|\wedge}^{\mathfrak{B}}d \in | \notin \mathcal{F}$ hold,
  in which case, by (3.52), (3.66) and (3.68), $M_{1|0}(d) \neq (m_n+1)$, and so $g(d) \neq ((n-1)|0)$, for, otherwise, we would have $\Bbbk(d) = (1|0)$, because $(2 \cdot (n-1)) \nleqslant (n-1)$, as $n \nleqslant 1$, in which case we would get $\ell(d) = 0$, and so would eventually get $M_{1|0}(d) = (m_n+1)$,

since $m < 0$, for no $m \in (m_n + 1)$. In particular, $\nabla_j^{\mathfrak{A}_n} g(d) = j$. Then, by (3.57), (3.58) and (3.63), we have $M_1(\nabla_j^{\mathfrak{B}} d) = \varnothing$ and, for all $m \in (m_n + 1)$, $(\partial_m^{\mathfrak{B}} \nabla_j^{\mathfrak{B}} d \in \mathcal{F}) \Leftrightarrow (m < j)$, in which case $g(\nabla_j^{\mathfrak{B}} d) = j = \nabla_j^{\mathfrak{A}_n} g(d)$.

- either of $\eth_{\vee|\wedge}^{\mathfrak{B}} d \notin | \in \mathcal{F}$ holds,
  in which case, by (3.68), $M_{0/1}(d) = ((\varnothing|(m_n+1))/((m_n+1)|\varnothing))$, and so $\Bbbk(d) = (1|0)$ and $\ell(d) = 0$, in which case $g(d) = ((n-1)|0)$, and so $\nabla_j^{\mathfrak{A}_n} g(d) = ((n-1)|0)$. Then, by both (3.59)|(3.61) and (3.60)|(3.62) as well as (3.68), we have $M_{0/1}(\nabla_j^{\mathfrak{B}} d) = ((\varnothing|(m_n+1))/((m_n+1)|\varnothing))$, in which case we get both $\Bbbk(\nabla_j^{\mathfrak{B}} d) = (1|0)$ and $\ell(\nabla_j^{\mathfrak{B}} d) = \min(m_n + 1) = 0$, and so eventually get $g(\nabla_j^{\mathfrak{B}} d) = ((n-1)|0) = \nabla_j^{\mathfrak{A}_n} g(d)$.

Furthermore, consider also any $e \in B$ and the following complementary cases:

- there is some $k \in 2$ such that both of $\langle d|e, \langle k, \ell(d|e)\rangle\rangle \in R$ hold.
  Consider the following complementary subcases:
  - $k = 0$,
    in which case $\partial_{m_n}^{\mathfrak{B}} {\sim}^{\mathfrak{B}} d \in \mathcal{F}$, and so, by (2.3) and (3.66), we have $\partial_{m_n}^{\mathfrak{B}} {\sim}^{\mathfrak{B}} (d \wedge^{\mathfrak{B}} e) \in \mathcal{F}$. Then, for each $m \in (m_n + 1)$, $(\partial_m^{\mathfrak{B}}(d|e) \in \mathcal{F}) \Leftrightarrow (m < \ell(d|e))$, in which case, by (3.65), $(\partial_m^{\mathfrak{B}}(d \wedge^{\mathfrak{B}} e) \in \mathcal{F}) \Leftrightarrow ((\partial_m^{\mathfrak{B}} d \in \mathcal{F}) \& (\partial_m^{\mathfrak{B}} e \in \mathcal{F})) \Leftrightarrow ((m < \ell(d)) \& (m < \ell(e))) \Leftrightarrow (((m + 1) \leqslant \ell(d)) \& ((m + 1) \leqslant \ell(e))) \Leftrightarrow ((m + 1) \leqslant \min(\ell(d), \ell(e))) \Leftrightarrow (m < \min(\ell(d), \ell(e)))$, and so $\langle d \wedge^{\mathfrak{B}} e, \langle 0, \min(\ell(d), \ell(e))\rangle\rangle \in R$. In this way, $g(d \wedge^{\mathfrak{B}} e) = \min(\ell(d), \ell(e)) = \min(g(d), g(e))$.
  - $k = 1$,
    in which case $\partial_{m_n}^{\mathfrak{B}}(d|e) \in \mathcal{F}$, and so, by (3.65), $\partial_{m_n}^{\mathfrak{B}}(d \wedge^{\mathfrak{B}} e) \in \mathcal{F}$. Then, for each $m \in (m_n + 1)$, $(\partial_m^{\mathfrak{B}} {\sim}^{\mathfrak{B}} (d|e) \in \mathcal{F}) \Leftrightarrow (m < \ell(d|e))$, in which case, by (2.3) and (3.66), $(\partial_m^{\mathfrak{B}} {\sim}^{\mathfrak{B}} (d \wedge^{\mathfrak{B}} e) \notin \mathcal{F}) \Leftrightarrow ((\partial_m^{\mathfrak{B}} {\sim}^{\mathfrak{B}} d \notin \mathcal{F}) \& (\partial_m^{\mathfrak{B}} {\sim}^{\mathfrak{B}} e \notin \mathcal{F})) \Leftrightarrow ((m \geqslant \ell(d)) \& (m \geqslant \ell(e))) \Leftrightarrow (m \geqslant \max(\ell(d), \ell(e)))$, and so $\langle d \wedge^{\mathfrak{B}} e, \langle 1, \max(\ell(d), \ell(e))\rangle\rangle \in R$. In this way, $g(d \wedge^{\mathfrak{B}} e) = (n - 1 - \max(\ell(d), \ell(e))) = \min(n - 1 - \ell(d), n - 1 - \ell(e)) = \min(g(d), g(e))$.
- there is some $\bar{f} \in \{d, e\}^2$ such that, for each $k \in 2$, $\langle f_k, \langle k, \ell(f_k)\rangle\rangle \notin R$,
  in which case $\langle f_k, \langle 1 - k, \ell(f_k)\rangle\rangle \in R$, and so $f_0 \neq f_1$ (in particular, $(\operatorname{img} \bar{f}) = \{d, e\}$). Then, $g(f_1) = \ell(f_1) \leqslant l_n \leqslant (n - 1 - l_n) \leqslant (n - 1 - \ell(f_0)) = g(f_0)$. And what is more, $\partial_{m_n}^{\mathfrak{B}} {\sim}^{\mathfrak{B}} f_1 \in \mathcal{F}$, in which case, by (2.3) and (3.66), $\partial_{m_n}^{\mathfrak{B}} {\sim}^{\mathfrak{B}} (f_0 \wedge^{\mathfrak{B}} f_1) \in \mathcal{F}$, while $\partial_{m_n}^{\mathfrak{B}} f_0 \in \mathcal{F}$, whereas, for each $m \in (m_n + 1)$, $(\partial_m^{\mathfrak{B}} f_1 \in \mathcal{F}) \Leftrightarrow (m < \ell(f_1))$, in which case, by (3.64) and (3.65), $(\partial_m^{\mathfrak{B}}(f_0 \wedge^{\mathfrak{B}} f_1) \in \mathcal{F}) \Leftrightarrow (\partial_m^{\mathfrak{B}} f_1 \in \mathcal{F}) \Leftrightarrow (m < \ell(f_1))$, and so $\langle f_0 \wedge^{\mathfrak{B}} f_1, \langle 0, \ell(f_1)\rangle\rangle \in R$. In this way,

$g(d \wedge^{\mathfrak{B}} e) = g(f_0 \wedge^{\mathfrak{B}} f_1) = \ell(f_1) = g(f_1) = \min(g(f_0), g(f_1)) = \min(g(d), g(e))$.

Thus, anyway, $g(d \wedge^{\mathfrak{B}} e) = \min(g(d), g(e)) = (g(d) \wedge^{\mathfrak{A}_n} g(e))$.

In this way, by (2.1) and (2.2), $g \in H \triangleq \hom(\mathfrak{B}, \mathfrak{A}_n)$.

Finally, we prove, by contradiction, that $g(a_0) \neq g(a_1)$. For suppose $g(a_0) = g(a_1)$, in which case, as $g \in H$, $g(b) = \max(g(a_0), g(a_1)) = \min(g(a_0), g(a_1)) = g(c)$. Consider any $k \in 2$ and any $m \in (m_n + 1)$. Assume either of $\partial^{\mathfrak{B}}_m(\sim^{\mathfrak{B}})^k(b|c) \notin \mathcal{F}$ holds. Then, by (3.64) with $\imath = m_n$, $\Bbbk(b|c) = k$, in which case $\ell(b|c) \leqslant m$, and so $\Bbbk(c|b) = \Bbbk(b|c) = k$, for otherwise, we would have $m_n \geqslant m \geqslant \ell(b|c) = (n-1-)^{\Bbbk(b|c)}g(b|c) = (n-1-(n-1-)^{\Bbbk(c|b)}g(c|b))) = (n-1-\ell(c|b)) \geqslant (n-1-l_n) \geqslant l_n \geqslant m_n$, in which case we would get $m_n = l_n = (n-1-l_n)$, and so would eventually get $m_n < (n-1-m_n) = m_n$. Hence, $\ell(c|b) = (n-1-)^{\Bbbk(c|b)}g(c|b) = (n-1-)^{\Bbbk(b|c)}g(b|c) = \ell(b|c) \leqslant m$. Therefore, $\partial^{\mathfrak{B}}_m(\sim^{\mathfrak{B}})^k(c|b) \notin \mathcal{F}$. Thus, $\partial^{\mathfrak{B}}_m(\sim^{\mathfrak{B}})^k b \in \mathcal{F}$ iff $\partial^{\mathfrak{B}}_m(\sim^{\mathfrak{B}})^k c \in \mathcal{F}$. In this way, by (3.67) and (3.68), $(b \leftrightarrow^{\mathfrak{B}} c) \in \mathcal{F}$, and so, by (3.25), we get $c \in \mathcal{F}$, for $b \in \mathcal{F}$. This contradiction shows that $g(a_0) \neq g(a_1)$.

Thus, $\hbar : B \to n^H, d \mapsto \langle h(d)\rangle_{h \in H}$ is an embedding of $\mathfrak{B}$ into $\mathfrak{A}_n^H$, and so $\mathfrak{B} \in \mathbf{QV}(\mathfrak{A}_n)$, as required. ⊣

Thus, the present subsection provides four essentially different insights into the equational substance of $\mathbf{QV}(\mathfrak{A}_n)$. In this connection, it is remarkable that, although different [quasi]equational axiomatizations of a given [quasi]variety are derivable from one another by means of congruence quasi-identities (cf., e.g., [13]) — that is, any [quasi-]identity, being satisfied in a [quasi]variety, is derivable from any [quasi]equational axiomatization of it endowed with congruence quasi-identities, such is not at all evident *immediately* for the above equational axiomatizations of $\mathbf{QV}(\mathfrak{A}_n)$. In this way, these are *practically* independent from one another and become good test samples for various systems of Automated Deduction in first-order universal Horn logic *with equality* (cf., e.g., [22]).

On the other hand, the present study would not be complete without investigating the four-valued case.

A *quadro-graded Kleene algebra* is any Σ_4-algebra, the $\Sigma_{\sim,01}$-reduct of which is a Kleene algebra and which satisfies the identities (3.38), (3.65), (3.67) and (3.68) but with $i = \imath = 1$ and $k = 0$ as well as the following additional identities:

$$\sim\partial_1 x_0 \approx \partial_1 \sim x_0, \tag{3.71}$$

$$(\sim x_0 \wedge \partial_1 x_0) \lessapprox x_0, \tag{3.72}$$

the variety of all them being denoted by GKA_4.

THEOREM 3.22. $\mathfrak{A}_4 \in \mathsf{GKA}_4$. *Conversely, any [finite] quadro-graded Kleene algebra is embeddable into a [finite] direct power of* $\mathfrak{A}_4$. *In particular,* GKA_4 *is the quasivariety generated by* $\mathfrak{A}_4$.

PROOF. The fact that $\mathfrak{A}_4 \in \mathsf{GKA}_4$ is immediate, with using (3.1) and (3.7). Conversely, consider any [finite] $\mathfrak{B} \in \mathsf{GKA}_4$ and any $\bar{a} \in (B^2 \setminus \Delta_B)$, in which case $b_0 \triangleq (a_0 \vee^{\mathfrak{B}} a_1) \nleqslant^{\mathfrak{B}} b_1 \triangleq (a_0 \wedge^{\mathfrak{B}} a_1) \leqslant^{\mathfrak{B}} b_0$, and so $b_0 \neq b_1$. Then, by Proposition 3.4 of [14], there is some $e \in \hom(\mathfrak{B}{\restriction}\Sigma_{\sim}, \mathfrak{K}_3)$ such that $e(b_0) \neq e(b_1)$, in which case $e(b_1) \leqslant e(b_0)$, while

$$e(c \vee^{\mathfrak{B}} {\sim}^{\mathfrak{B}} c) \in \{1, 2\}, \tag{3.73}$$

for all $c \in B$, whereas, since $\{1, 2\}$ is a prime filter of $\mathfrak{K}_3$, $\mathcal{F} \triangleq e^{-1}[\{1, 2\}]$ is that of $\mathfrak{B}$. Consider the following complementary cases:

- $e(b_1) = 0$,
  in which case $e(b_0) \neq e(b_1) = 0$, and so $b_0 \in \mathcal{F} \not\ni b_1$.
- $e(b_1) \neq 0$,
  in which case, as $e(b_1) < e(b_0)$, we have both $e(b_1) = 1$ and $e(b_0) = 2$, and so we get both $e({\sim}^{\mathfrak{B}} b_1) = 1$ and $e({\sim}^{\mathfrak{B}} b_0) = 0$. Then, ${\sim}^{\mathfrak{B}} b_1 \in \mathcal{F} \not\ni {\sim}^{\mathfrak{B}} b_0$.

Thus, in any case, there is some $k \in 2$ such that $({\sim}^{\mathfrak{B}})^k b_k \in \mathcal{F} \not\ni ({\sim}^{\mathfrak{B}})^k b_{1-k}$. In particular, $\varnothing \neq \mathcal{F} \neq B$, in which case $\top^{\mathfrak{B}} \in \mathcal{F} \not\ni \bot^{\mathfrak{B}}$, and so

$$(c \in \mathcal{F}) \Leftrightarrow ({\sim}^{\mathfrak{B}} c \notin \mathcal{F}), \tag{3.74}$$

for all $c \in \flat^{\mathfrak{B}}$. Let $f \triangleq ((\mathcal{F} \times \{1\}) \cup ((B \setminus \mathcal{F}) \times \{0\}))$ be the characteristic function of $\mathcal{F} \subseteq B$ in B and $g : B \to 4, d \mapsto (f(d) \cdot (1 + (f(\partial_1^{\mathfrak{B}} d) \cdot (2 - f({\sim}^{\mathfrak{B}} d))))$. Then, $g(\bot^{\mathfrak{B}}) = 0 = \bot^{\mathfrak{A}_4}$. Likewise, by (2.6) and (3.38), $g(\top^{\mathfrak{B}}) = 3 = \top^{\mathfrak{A}_4}$. Now, consider any $c \in B$ and the following exhaustive cases:

- $g(c) = 0$.
  Then, $f(c) = 0$, in which case, by (3.73)/(2.1), $f({\sim}^{\mathfrak{B}}({\sim}^{\mathfrak{B}})^{0/1} c) = (1/0)$, and so, by (3.72), $f(\partial_1^{\mathfrak{B}} c) = 0$. In particular, $g(\partial_1^{\mathfrak{B}} c) = 0$. And what is more, by (3.68), (3.71) and (3.74), $f(\partial_1^{\mathfrak{B}} {\sim}^{\mathfrak{B}} c) = 1$. Hence, $g({\sim}^{\mathfrak{B}} c) = 3$.
- $g(c) = 1$.
  Then, $f(c) = 1$, while $f(\partial_1^{\mathfrak{B}} c) = 0$, in which case, by (3.68), (3.71) and (3.74), $f(\partial_1^{\mathfrak{B}} {\sim}^{\mathfrak{B}} c) = 1$, and so, by (2.1) /"and (3.72)", $f({\sim}^{\mathfrak{B}} ({\sim}^{\mathfrak{B}})^{1/0} c) = 1$. Hence, $g(\partial_1^{\mathfrak{B}} c) = 0$ and $g({\sim}^{\mathfrak{B}} c) = 2$.
- $g(c) = 2$.
  Then, $f(c) = 1$, while $f(\partial_1^{\mathfrak{B}} c) = 1$, whereas $f({\sim}^{\mathfrak{B}} c) = 1$, in which case, by (3.67), $f(\partial_1^{\mathfrak{B}} \partial_1^{\mathfrak{B}} c) = 1$, while, by (3.68) and (3.71), $f(\partial_1^{\mathfrak{B}} {\sim}^{\mathfrak{B}} c) = f({\sim}^{\mathfrak{B}} \partial_1^{\mathfrak{B}} c) = 0$, and so $g(\partial_1^{\mathfrak{B}} c) = 3$ and $g({\sim}^{\mathfrak{B}} c) = 1$.

- $g(c) = 3$.
  Then, $f(c) = 1$, while $f(\partial_1^{\mathfrak{B}} c) = 1$, whereas $f(\sim^{\mathfrak{B}} c) = 0$, in which case, by (3.67), $f(\partial_1^{\mathfrak{B}} \partial_1^{\mathfrak{B}} c) = 1$, while, by (3.68), $f(\sim^{\mathfrak{B}} \partial_1^{\mathfrak{B}} c) = 0$, and so $g(\partial_1^{\mathfrak{B}} c) = 3$ and $g(\sim^{\mathfrak{B}} c) = 0$.

In this way, $g(\wr^{\mathfrak{B}} c) = \wr^{\mathfrak{A}_4} g(c)$, for every $\wr \in (\Sigma_4 \setminus \Sigma_{+,01}) = \{\sim, \partial_1\}$, for $N_{4-} = \{1\}$. Finally, consider also any $d \in B$. Then, $f(c \wedge^{\mathfrak{B}} d) = \min(f(c), f(d))$, while, by (3.65), $f(\partial_1^{\mathfrak{B}}(c \wedge^{\mathfrak{B}} d)) = \min(f(\partial_1^{\mathfrak{B}} c), f(\partial_1^{\mathfrak{B}} d))$, whereas, by (2.3), $f(\sim^{\mathfrak{B}}(c \wedge^{\mathfrak{B}} d)) = \max(f(\sim^{\mathfrak{B}} c), f(\sim_1^{\mathfrak{B}} d))$. Therefore, $g((c \wedge^{\mathfrak{B}} d) = (\min(f(c), f(d)) \cdot (1 + (\min(f(\partial_1^{\mathfrak{B}} c), f(\partial_1^{\mathfrak{B}} d)) \cdot (2 - \max(f(\sim^{\mathfrak{B}} c), f(\sim_1^{\mathfrak{B}} d)))))$. Consider the following complementary cases:

- $\min(f(c), f(d)) = 0$.
  Then, either of $f(c|d) = 0$ holds, in which case $g(c|d) = 0 \leqslant g(d|c)$, and so $g(c \wedge^{\mathfrak{B}} d) = 0 = \min(g(c), g(d))$.
- $\min(f(c), f(d)) = 1$.
  Then, both $f(c|d) = 1$ hold, in which case $g(c|d) \geqslant 1 \leqslant g(c \wedge^{\mathfrak{B}} d)$. Consider the following complementary subcases:
  - $\min(f(\partial_1^{\mathfrak{B}} c), f(\partial_1^{\mathfrak{B}} d)) = 0$.
    Then, either of $f(\partial_1^{\mathfrak{B}}(c/d)) = 0$ holds, in which case $g(c/d) = 1 \leqslant g(d/c)$, and so $g(c \wedge^{\mathfrak{B}} d) = 1 = \min(g(c), g(d))$.
  - $\min(f(\partial_1^{\mathfrak{B}} c), f(\partial_1^{\mathfrak{B}} d)) = 1$.
    Then, both $f(\partial_1^{\mathfrak{B}}(c/d)) = 1$ hold, in which case $g(c/d) \geqslant 2 \leqslant g(c \wedge^{\mathfrak{B}} d)$. Consider the following complementary subsubcases:
    * $\max(f(\sim^{\mathfrak{B}} c), f(\sim^{\mathfrak{B}} d)) = 1$.
      Then, either of $f(\sim^{\mathfrak{B}}(c\|d)) = 1$ holds, in which case $g(c\|d) = 2 \leqslant g(d\|c)$, and so $g(c \wedge^{\mathfrak{B}} d) = 2 = \min(g(c), g(d))$.
    * $\max(f(\sim^{\mathfrak{B}} c), f(\sim^{\mathfrak{B}} d)) = 0$.
      Then, both $f(\sim^{\mathfrak{B}}(c\|d)) = 0$ hold, in which case $3 \geqslant g(c\|d) \geqslant 3 \leqslant g(c \wedge^{\mathfrak{B}} d) \leqslant 3$, and so $g(c \wedge^{\mathfrak{B}} d) = 3 = \min(g(c), g(d))$.

Thus, anyway, $g(c \wedge^{\mathfrak{B}} d) = \min(g(c), g(d)) = (g(c) \wedge^{\mathfrak{A}_4} g(d))$.

In this way, by (2.1) and (2.2), $H \triangleq \hom(\mathfrak{B}, \mathfrak{A}_4) \ni g$ [is finite]. Moreover, $f((\sim^{\mathfrak{B}})^k b_{(1-)k}) = (1(-1))$, in which case we have $g((\sim^{\mathfrak{B}})^k b_{1-k}) = 0 \neq g((\sim^{\mathfrak{B}})^k b_k)$, and so, as $g \in H$, get $(3-)^k g(b_0) \neq (3-)^k g(b_1)$. Then, $\max(g(a_0), g(a_1)) = g(b_0) \neq g(b_1) = \min(g(a_0), g(a_1))$, in which case $g(a_0) \neq g(a_1)$, and so $\hbar : B \to n^H, d \mapsto \langle h(d) \rangle_{h \in H}$ is an embedding of $\mathfrak{B}$ into $\mathfrak{A}_4^H$, as required. ⊣

REFERENCES

[1] F. G. Asenjo and J. Tamburino, *Logic of antinomies*, ***Notre Dame Journal of Formal Logic***, vol. 16 (1975), pp. 272–278.

[2] R. Balbes and P. Dwinger, ***Distributive Lattices***, University of Missouri Press, Columbia (Missouri), 1974.

[3] N. D. Belnap, Jr, *A useful four-valued logic*, ***Modern uses of multiple-valued logic*** (J. M. Dunn and G. Epstein, editors), D. Reidel Publishing Company, Dordrecht, 1977, pp. 8–37.

[4] K. Hałkowska and A. Zajac, *O pewnym, trójwartościowym systemie rachunku zdań*, ***Acta Universitatis Wratislaviensis. Prace Filozoficzne***, vol. 57 (1988), pp. 41–49.

[5] J. Łoś and R. Suszko, *Remarks on sentential logics*, ***Indagationes Mathematicae***, vol. 20 (1958), pp. 177–183.

[6] A. I. Mal'cev, ***Algebraic systems***, Springer Verlag, New York, 1965.

[7] E. Mendelson, ***Introduction to mathematical logic***, 2nd ed., D. Van Nostrand Company, New York, 1979.

[8] C. S. Peirce, *On the Algebra of Logic: A Contribution to the Philosophy of Notation*, ***American Journal of Mathematics***, vol. 7 (1885), pp. 180–202.

[9] A. G. Pinus, ***Congruence-modular varieties of algebras***, Irkutsk University Press, Irkutsk, 1986, In Russian.

[10] G. Priest, *The logic of paradox*, ***Journal of Philosophical Logic***, vol. 8 (1979), pp. 219–241.

[11] A. P. Pynko, *Characterizing Belnap's logic via De Morgan's laws*, ***Mathematical Logic Quarterly***, vol. 41 (1995), no. 4, pp. 442–454.

[12] ———, *On Priest's logic of paradox*, ***Journal of Applied Non-Classical Logics***, vol. 5 (1995), no. 2, pp. 219–225.

[13] ———, *Definitional equivalence and algebraizability of generalized logical systems*, ***Annals of Pure and Applied Logic***, vol. 98 (1999), pp. 1–68.

[14] ———, *Implicational classes of De Morgan lattices*, ***Discrete mathematics***, vol. 205 (1999), pp. 171–181.

[15] ———, *Subprevarieties versus extensions. Application to the logic of paradox*, ***The Journal of Symbolic Logic***, vol. 65 (2000), no. 2, pp. 756–766.

[16] ———, *Extensions of Hałkowska-Zajac's three-valued paraconsistent logic*, ***Archive for Mathematical Logic***, vol. 41 (2002), pp. 299–307.

[17] ———, *Sequential calculi for many-valued logics with equality determinant*, ***Bulletin of the Section of Logic***, vol. 33 (2004), no. 1, pp. 23–32.

[18] ———, *A relative interpolation theorem for infinitary universal Horn logic and its applications*, ***Archive for Mathematical Logic***, vol. 45 (2006), pp. 267–305.

[19] ———, *Subquasivarieties of implicative locally-finite quasivarieties*, ***Mathematical Logic Quarterly***, vol. 56 (2010), no. 6, pp. 643–658.

[20] ———, *Four-valued expansions of Dunn-Belnap's logic (I): Basic characterizations*, ***Bulletin of the Section of Logic***, vol. 49 (2020), no. 4, pp. 401–437.

[21] ———, *Finite Hilbert-style axiomatizations of disjunctive and implicative finitely-valued logics with equality determinant*, (2021), Submitted.

[22] A. Robinson and A. Voronkov (editors), ***Handbook of automated reasoning***, Elsevier Science Publishers, 2001.

[23] J.A. Robinson, *A machine-oriented logic based on the resolution principle*, ***Journal of Association of Computing Machinery***, vol. 12 (1965), pp. 23–41.

[24] A. M. Sette, *On the propositional calculus P^1*, ***Mathematica Japonica***, vol. 18 (1973), pp. 173–180.

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