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Abstract. This article presents the study of fuzzy multi-objective linear fractional programming (FMOLFP) problem by using operators. In this approach, the FMOLFP problem is transformed to fuzzy multi-objective linear fractional programming (FMOLFP) problem by using suitable transformation [1]. The reduced problem is then solve by using min operator and average operator model and hence find out the solutions of the problem. One numerical example is presented to demonstrate the approach.
Keywords: Multi-objective linear programming, Multi-objective linear fractional programming, Fuzzy set theory, Min operator, Average operator, Fuzzy efficient solutions.

1 Introduction

Linear fractional programming problem was studied extensively in the middle of 1960s and early 1970s of the last century [1]. In many practical applications like stock problems, ore-blending problems, shipping schedule problems, optimal policy for a Markovian chains, sensitivity of linear programming (LP) problem, optimization of ratio criterion gives more insight into the situations than the optimization of each criterion [9].
Multi-objective linear fractional programming (MOLFP) problem is used to describe the problem associated with multiple objectives where objective functions are written in fractional formulas. These type of problems has been used in production planning, inventory management, financial and banking sector, etc.

The concept of 'Decision making in fuzzy environment' was first studied by Bellman and Zadeh [2]. Zimmermann [4] first proposed the fuzzy linear programming (FLP) problem. Charnes and Cooper [1] solved a programming problem with linear fractional functionals by resolving it into two linear programming (LP) problem. By suitable transformation, Chakraborty and Gupta [18] have transformed MOLFP problem to formulate an equivalent MOLP problem under fuzzy set theoretic approach. Furthere more, there are a few studies [6],[24], [26],[27] on MOLFP problem. The min-operator model is proposed by Zimmermann [4] to solve multi-objective linear programming (MOLP) problem. Luhandjula [5] proved some properties of min operator but the solution obtained by min operator doesn't give compensatory and efficient solutions [23],[14]. To overcome this difficulty, Lee and Li [23] have proposed two-phase approach to get more efficient and satisfactory result. Guu et.al [13] have applied two phase approach in MOLFP problem while Chen and Chou [12] proposed a fuzzy approach to integrate the min operator, average operator and two-phase methods. In 1997, Guu and Wu [14] have solved two phase approach with positive weighted coefficients, not necessarily equal, for solving the MOLP problems gives an efficient solution and they [16] proposed a two-phase method to improve the solution yielded by the max-min operator when solving the linear programming with imprecision parameters. Zimmermann and Zysno [28] observed from an experiment that most of the real world problems are neither non-compensatory (min-operator) nor full compensatory (average operator). In 2001, Wu and Guu have presented a compromise model between non-compensatory (min-operator) nor full compensatory (average operator) for obtaining fuzzy efficient solution of MOLP problem. To solve fuzzy linear programming problems (FLPP), Werner's [7] proposed membership functions for the fuzzy objective and applied the concept of max-min operator [4]. In this paper, we have studied MOLFP problem under fuzzy environment where MOLFP problem is converted to MOLPP by suitable transformation. Min-operator and average operator model are used to solve MOLP problem and hence find out the solution of the problem.
The paper is organised as follows:-
Section 2 outlines the definitions of LFP and MOLFP. In section 3, we have discussed the transformation of MOLFP to MOLP, the min operator and average operator models are used in transformed MOLP and hence find out the solutions. Section 4 demonstrates one numerical example to illustrate our approach. Section 5 discusses the conclusions of this paper.

2 Definitions and Preliminaries

Definition 2.1. [25] Linear Fractional Programming- The general format of linear fractional programming (LFP) may be written as:

$$Max\, Z(x) = \tfrac{cx+\alpha}{dx+\beta},$$

subject to the constraints:

$$x \in S = \{x|Ax = b, x \geq 0\}$$

where $A \in R^{m\times n}$, $b \in R^m$, $x \in R^n$, $c, d \in R^n$, $\alpha, \beta \in R$ and S is a non-empty and bounded set.

Definition 2.2. [25] Multi-objective Linear Fractional Programming Problem- The general format of a multi-objective linear fractional programming problem which is stated as follows-

$$\text{Max Z(x)}=\{Z_1(x), Z_2(x), ..., Z_n(x)\}$$

subject to the constraints:

$$x \in S = \{x \in R^n : Ax \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, x \geq 0\}$$
$$\text{with } b \in R^m, A \in R^{m\times n}$$

$$\text{and} \quad Z_i(x) = \tfrac{c_i x+\alpha_i}{d_i x+\beta_i} = \tfrac{N_i(x)}{D_i(x)};$$

where c_i, $d_i \in R^n$ and α_i, β_i are constants and S $\neq \phi$.

3 Methodology

Transformation of FMOLFP problem to FMOLP problem:
Consider the following fuzzy multi-objective linear fractional programming problem as follows-

$$\text{Max } Z(x) = \{Z_1(x), Z_2(x), ..., Z_n(x)\}$$

subject to the constraints:

$$x \in S = \{x \in R^n : (Ax)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \check{b_j}, x \geq 0\}..........(1)$$

$$\text{with } b \in R^m, A \in R^{m\times n} \text{ and}$$

$$Z_i(x) = \tfrac{c_i x+\alpha_i}{d_i x+\beta_i} = \tfrac{N_i(x)}{D_i(x)};$$

for all i=1,2,...,n and j=1,2,...,m ;

where c_i, $d_i \in R^n$ and α_i, β_i are constants.

By using Charne's and Cooper method [1], substituting y=tx in (1), we get,

$$Max \quad f_i(y,t) = c_i y + \alpha_i t$$

subject to the constraints:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} 0,$$
$$d_i y + \beta_i t = 1$$

where y, $t \geq 0$.

The above FMOLFP problem is equivalent to the following FMOLP problem as follows:

$$Max\, f_i(y,t) = c_i y + \alpha_i t$$

subject to the constraint:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} 0, ...(2)$$

$$d_i y + \beta_i t = 1$$

where y, $t \geq 0$

where $f_i(y,t)$, i=1,2,.....,n are affine functions, fuzzy resources $\check{p}_j$ is in $[p_j, p_j + q_j]$ with given q_j (without loss of generality we assume that $0 < q_j < \infty$) for each j.

Werners [7],[8] proposed a max-min operator method for crisp objective function of problem (2), which is similar to Zimmermann [4]. The possible range $[f_i^0, f_i^1]$ for the i^{th} objective function can be obtained as follows-

$$f_i^0 = Max f_i$$

subject to the constraints:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} p_j, ...(3)$$

$$d_i y + \beta_i t = p_j$$
where y, $t \geq 0$

and

$$f_i^1 = Max f_i$$

subject to the constraints:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} p_j + q_j, ...(4)$$

$$d_i y + \beta_i t = p_j + q_j$$
where y,$t \geq 0$.

A non-decreasing linear membership function for the i^{th} objective function is defined as follows:

$$\mu_{0i}(y,t) = \begin{cases} 1 & , f_i(y,t) > f_i^1 \\ \frac{f_i(y,t) - f_i^0}{f_i^1 - f_i^0} & , f_i^0 \leq f_i(y,t) \leq f_i^1 ....(5) \\ 0 & , f_i(y,t) < f_i^0 \end{cases}$$

A non-increasing linear membership function for the j^{th} fuzzy constraint is defined as follows:

$$\mu_j(y,t) = \begin{cases} 1 & , (Ay - bt)_j < p_j \\ \frac{(p_j + q_j) - (Ay - bt)_j}{q_j} & , p_j \leq (Ay - bt)_j \leq p_j + q_j ....(6) \\ 0 & , (Ay - bt)_j > p_j + q_j \end{cases}$$

When all the membership functions corresponding to objective functions and constraints are known, the FMOLFP problem is solved by following approach-
Let λ_i be the minimum acceptability of the i^{th} objective and λ_{n+j} be the minimum acceptability of the j^{th} constraint.

$$\text{Let } \lambda = min\{\lambda_i, \lambda_{n+j}\} \qquad \forall i, j$$

Using the max-min principle of Bellman and Zadeh [2] and introducing the variable λ adopts the following formulation [4]:

$$Max \lambda$$

subject to the constraints :

$$1 \geq \mu_{0i}(y,t) \geq \lambda \geq 0, \forall i = 1, 2, ..., n$$

$$1 \geq \mu_j(y,t) \geq \lambda \geq 0, \forall j = 1, 2, ..., m.......(7)$$

$$\lambda \in [0,1], y, t \geq 0.$$

Solving the model (7), one optimal value λ^* can be obtained. In fact, this λ^* denotes that the satisfaction level for all membership functions can simultaneously obtain. Further, let us assume that the membership functions of all objective and constraint are equally important. The model (2) can be solved to the following average operator model:

$$Max \quad \lambda^{**} = \frac{1}{n+m} \sum_{k=1}^{n+m} \lambda_k,$$

subject to the constraints:

$$\begin{aligned} &1 \geq \mu_{0i}(y,t) \geq \lambda_i \geq 0,\ \forall i = 1,2,......., n \\ &1 \geq \mu_j(y,t) \geq \lambda_{n+j} \geq 0,\ \forall j = 1,2,......., m \quad (8) \\ &\lambda \in [0,1],\ y,t \geq 0. \end{aligned}$$

The optimal value λ^{**} represents the total amount of all membership functions.

To obtain Fuzzy efficient solutions:

Definition 3.1 [22]: A decision plan $x^* \in S$ is said to be a Pareto-optimal solution to the MOLP problem (2) iff there doesn't exist another $x \in S$ such that

$$f_k(x) \leq f_k(x^*) \text{ for all k (k=1,2,....,N)}$$
$$\text{and } f_l(x) < f_l(x^*) \text{ for atleast one l.}$$

Definition 3.2 [7],[8]: A decision plan $x^* \in S$ is said to be fuzzy-efficient solution to the FMOLFP problem (7) if and only if there does not exist another $x \in S$ such that

$$\mu_k(f_k(x)) \geq \mu_k(f_k(x^*)) \text{ for all k (k=1,2,....,N)}$$
$$\text{and } \mu_l(f_l(x)) > \mu_l(f_l(x^*) \text{ for atleast one l.}$$

It is obvious that any fuzzy-efficient solution x^* to FMOLFP problem such that $f_i(y,t) \in (p_j, p_j + q_j)$ for all j, is a pareto-optimal solution to the FMOLFP problem (1). But if membership degree is 1, the fuzzy-efficiency does not gurantee pareto-optimality which is given in the following observation-

Observation 1 [22]: Let x^* be a fuzzy-efficient solution to the FMOLFP problem (1) such that $\mu_l(f_l(x^*)) = 1$ for some l, i.e., $f_l(x^*) \leq p_l$, then it could be the case that x^* is not a pareto-optimal solution. This is due to the fact that on the left of p_l the membership function μ_l is constantly equal to 1.

4 Numerical Example

Consider a FMOLFP problem with two objective functions as follows: [18]

$$\text{Max } (Z_1(x) = \tfrac{6x_1+5x_2}{2x_1+7}, Z_2(x) = \tfrac{2x_1+3x_2}{x_1+x_2+7})$$

subject to the constraints:

$$\begin{aligned} x_1 + 2x_2 &\leq 3, \\ 3x_1 + 2x_2 &\leq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution:
Here, $Z_1(x), Z_2(x) \geq 0$ for some x in the feasible region. The above MOLFP problem is equivalent to the following MOLP problem-

$$\text{Max}(f_1(y,t) = 6y_1 + 5y_2, f_2(y,t) = 2y_1 + 3y_2)$$

subject to the constraints:

$$\begin{aligned} g_1(y,t) &= y_1 + 2y_2 - 3t \leq 0, \\ g_2(y,t) &= 3y_1 + 2y_2 - 6t \leq 0, \\ g_3(y,t) &= 2y_1 + 7t = 1, \\ g_4(y,t) &= y_1 + y_2 + 7t = 1, \\ y_1, y_2, t &\geq 0 \end{aligned}$$

where the fuzzy resources with the corresponding maximal tolerances are $p_1 = 0.5$, $p_2 = 0.15$, $p_3 = 5$, $p_4 = 15$. Solving by LP package, the range of the objective function are as follows:

$$[f_1^0, f_1^1] = [1.21, 6.60]$$
$$\text{and } [f_2^0, f_2^1] = [0.55, 3.96].$$

The membership function of the two objective functions are defined as follows:

$$\mu_{01}(y,t) = \begin{cases} 1 & , if \quad f_1(y,t) > 6.60 \\ \frac{f_1(y,t)-1.21}{5.39} & , if \quad 1.21 \leq f_1(y,t) \leq 6.60 \\ 0 & , if \quad f_1(y,t) < 1.21 \end{cases}$$

and

$$\mu_{02}(y,t) = \begin{cases} 1 & , if f_2(y,t) > 3.96 \\ \frac{f_2(y,t)-0.55}{3.41} & , if \quad 0.55 \leq f_2(y,t) \leq 3.96 \\ 0 & , if \quad f_2(y,t) < 0.55 \end{cases}$$

For each of fuzzy constraints, the non-increasing linear membership functions are written as follows:

$$\mu_1(y,t) = \begin{cases} 1 & , if \quad g_1(y,t) < 0 \\ \frac{0.5-g_1(y,t)}{0.5} & , if \quad 0 \leq g_1(y,t) \leq 0.5 \\ 0 & , if \quad g_1(y,t) > 0.5 \end{cases}$$

$$\mu_2(y,t) = \begin{cases} 1 & , if \quad g_2(y,t) < 0 \\ \frac{0.15-g_2(y,t)}{0.15} & , if \quad 0 \leq g_2(y,t) \leq 0.15 \\ 0 & , if \quad g_2(y,t) > 0.15 \end{cases}$$

$$\mu_3(y,t) = \begin{cases} 1 & , if \quad g_3(y,t) < 1 \\ \frac{5-g_3(y,t)}{4} & , if \quad 1 \leq g_3(y,t) \leq 5 \\ 0 & , if \quad g_3(y,t) > 5 \end{cases}$$

$$\mu_4(y,t) = \begin{cases} 1 & , if \quad g_4(y,t) < 1 \\ \frac{15-g_4(y,t)}{14} & , if \quad 1 \leq g_4(y,t) \leq 15 \\ 0 & , if \quad g_4(y,t) > 15 \end{cases}$$

When the membership functions of each objective and fuzzy constraints are determined, the Phase 1 of two-phase approach as the same with min operator will be as follows:

$$\text{Max } \lambda$$

subject to the constraints:

$$\begin{gathered} 6y_1 + 5y_2 \leq 6.60, \\ 6y_1 + 5y_2 - 5.39\lambda \geq 1.21, \\ 2y_1 + 3y_2 \leq 3.96, \\ 2y_1 + 3y_2 - 3.41\lambda \geq 0.55, \\ y_1 + 2y_2 - 3t \geq 0, \\ y_1 + 2y_2 - 3t + 0.5\lambda \leq 0.5, \\ 3y_1 + 2y_2 - 6t \geq 0, \\ 3y_1 + 2y_2 - 6t + 0.15\lambda \leq 0.15, \\ 2y_1 + 7t \geq 1, \\ 2y_1 + 7t + 4\lambda \leq 5, \\ y_1 + y_2 + 7t \geq 1, \\ y_1 + y_2 + 7t + 14\lambda \leq 15, \\ y_1, y_2, t \geq 0, \text{ and } \lambda \in [0,1]. \end{gathered}$$

Solving we get, the optimal solution is-

$$y^* = (0.38, 0.46, 0.34)$$

The optimal value and membership function for all objectives and constraints are as follows:

$$\begin{gathered} \lambda^* = 0.46,\ f_1(y^*) = 4.58,\ f_2(y^*) = 2.14, \\ \mu_{01}(y^*) = 0.63,\ \mu_{02}(y^*) = 0.47 = \mu_3(y^*), \\ \mu_1(y^*) = 0.44,\ \mu_2(y^*) = 0.87,\ \mu_4(y^*) = 0.84.........(9) \\ \therefore x^* = (1.12, 1.35). \\ \text{Hence, } Z_1(x^*) = 1.46,\ Z_2(x^*) = 0.66. \end{gathered}$$

Applying the results y^* generated from phase 1, the second phase of two-phase approach as model (8) is to solve the following problem:

$$Max \quad \bar{\lambda} = \frac{1}{6}\sum_{k=1}^{6} \lambda_k$$

subject to the constraints:

$$\begin{gathered}
\lambda_k \geq 0.46, \\
6y_1 + 5y_2 \leq 6.60, \\
6y_1 + 5y_2 - 5.39\lambda \geq 1.21, \\
2y_1 + 3y_2 \leq 3.96, \\
2y_1 + 3y_2 - 3.41\lambda \geq 0.55, \\
y_1 + 2y_2 - 3t \geq 0, \\
y_1 + 2y_2 - 3t + 0.5\lambda \leq 0.5, \\
3y_1 + 2y_2 - 6t \geq 0, \\
3y_1 + 2y_2 - 6t + 0.15\lambda \leq 0.15, \\
2y_1 + 7t \geq 1, \\
2y_1 + 7t + 4\lambda \leq 5, \\
y_1 + y_2 + 7t \geq 1, \\
y_1 + y_2 + 7t + 14\lambda \leq 15, \\
y_1, y_2, t \geq 0, \text{ and } \lambda \in [0, 1].
\end{gathered}$$

Solving the above problem, the optimal solution is

$$y^{**} = (0.13, 0.13, 0.11).$$

The value of the objective functions and membership functions are as follows:

$$\begin{gathered}
\bar{\lambda} = 0.46,\ f_1(y^{**}) = 1.43,\ f_2(y^{**}) = 0.65, \\
\mu_{01}(y^{**}) = 0.04,\ \mu_{02}(y^{**}) = 0.03\ , \\
\mu_1(y^{**}) = 0.88,\ \mu_2(y^{**}) = 1,\ \mu_3(y^{**}) = 0.99 = \mu_4(y^{**}) \text{.........(10)} \\
\therefore x^{**} = (1.18, 1.18). \\
\text{Hence, } Z_1(x^{**}) = 1.39,\ Z_2(x^{**}) = 0.56.
\end{gathered}$$

Let us compare the two-phase approach results (10) with the solution (9) obtained by min-operator. It is observed that the membership function $\mu_2(y^{**}) = 1$ which is larger than $\mu_2(y^*) = 0.87$, which means that the two phase approach really obtains fuzzy-efficient solution and improves the min-operator's solution. But by observation 1 [22], we see that a fuzzy efficient solution is is not pareto-optimal solution.

5 Conclusions

In this paper, we have studied FMOLFP by using min-operator and average operator model. By using Guu and Wu [13] approach we have found out the fuzzy-efficient solution of the FMOLFP problem. We have shown that the membership degree in average operator model gives fuzzy-efficient solution and improves the min operator solution. With the help of this approach, one numerical example is solved and from that we see that the solution obtained by average operator is fuzzy-efficient.

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