Generalized Graph Complexes: the Splitting into a Complete Graph

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Abstract

Graph complexes are graded vector spaces of formal linear combinations of isomorphism classes of some kind of graphs, with the differential defined by vertex splitting (or, dually, edge contraction).

In this paper a generalization, where vertex splitting (i.e. insertion of the complete graph on two vertices) is replaced by insertion of a complete graph, is introduced. Basic properties are shown.

1 "Traditional" graph complexes

1.1 Explanations for the IHX-relation

If we consider a 3-regular graph G (every vertex has valency 3) and we collapse an edge e then the obtained graph G_e has exactly one 4-valent vertex. As the figure below shows, there are exactly two other graphs whose edge collapse leads to the graph G_e.

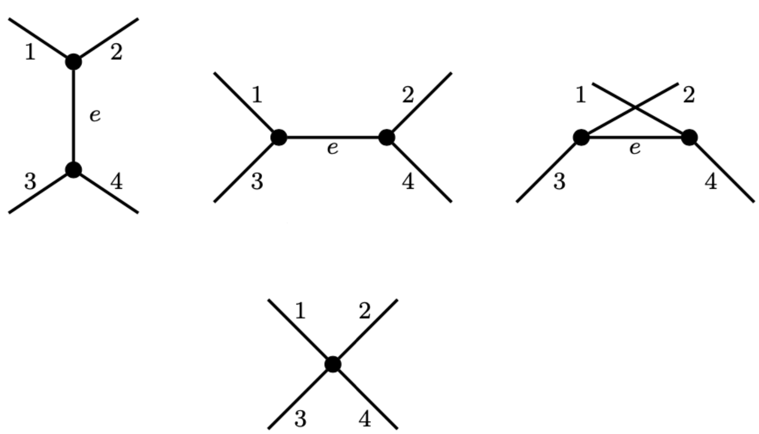

Figure 1: The numbers represent the four parts of the graph that get connected at e

The only axiom in the definition of a Lie algebra, in addition to the bilinearity and skew-symmetry of the Lie bracket, is the Jacobi identity. Picturing the Lie bracket as a rooted Y-tree with two inputs and one output, the Jacobi identity can be encoded by the next figure:

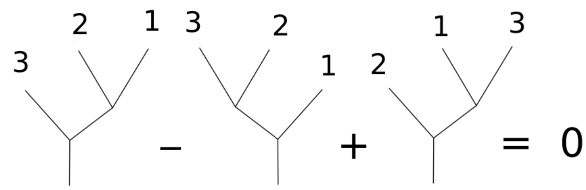

Figure 2: The Jacobi identity

Labeling the root, Figure 2 may be redrawn with the position of the labeled univalent vertices fixed as follows:

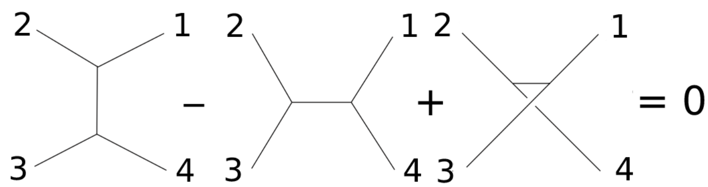

Figure 3: The IHX-relation (because of its appearance)

The above explains results of Maxim Kontsevich $[1-2]$ in his study of the formality conjecture, which introduced the most basic graph complexes. These complexes are formed by vacuum Feynman diagrams of a topological field theory (alternatively, it governs the deformation theory of E_n operads in algebraic topology [3]).

Dirk Kreimer et al. [4] showed how gauge theory amplitudes can be generated using only a scalar field theory with cubic interaction, i.e. all graphs relevant in gauge theory can be generated from the set of all 3-regular graphs by means of operators that label edges and cycles.

1.2 Definition

Let G be a connected graph. Let $V(G), E(G)$ denote its set of vertices, and edges, and denote by

h_G : the number of loops, or genus, of G,

e_G : the number of edges of G,

$deg_N G = e_G - N h_G$: the degree of G for a fixed non negative integer N.

An orientation on G is an element

$$\eta \in (\bigwedge^{e_G} \mathbb{Z}^{E(G)})^{\times}.$$

If the edges of G are denoted by $e_1, \ldots, e_n$, where $n = e_G$, then an orientation is equal to either $e_1 \wedge \cdots \wedge e_n$ or its negative. Thus, an orientation is simply an ordering of the edges up to the action of even permutations (interchanging any two edges reverses the orientation). As proved in [5], this definition of orientation is equivalent to the definition given by Maxim Kontsevich.

Let GC_n be the abstract vector space over $\mathbb{Q}$ spanned by equivalence classes of pairs (G, o), where G is a connected non empty graph with n vertices and o is the orientation of G.

$$GC = \bigoplus_n GC_n.$$

The differential $d : GC_n \longrightarrow GC_{n-1}$ is given by

$$d(G, o) = \sum_{e \in E(G)} (G/e, o_{G/e}),$$

where G/e is the graph obtained from G by contracting the edge e and $o_{G/e}$ is the induced orientation obtained by moving the edge e in the first place and then removing it.

By an **edge contraction**, it means an operation of the following form (see also Figure 1):

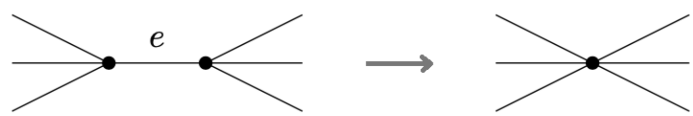

Figure 4: An edge contraction

The complex (GC, d^*), where d^* is the dual differential, carries the structure of a dg Lie algebra. The differential d^* is defined by

$$d^*(G) = \sum_{v \in V(G)} \mathrm{split}(G, v),$$

where $\mathrm{split}(G, v)$ is the operation that replaces the vertex v by two vertices connected by an edge and reconnects edges that were connected to the vertex v to the two new vertices in all possible ways.

A feature characterizing the cohomology of GC is that it depends on the parity of N in the degree of a graph G [3, 6].

The cohomologies of the various graph complexes are highly related. Obviously, the graph complexes with disconnected graphs are symmetric products of the complexes of connected graphs. Moreover, it was shown in [3] that adding the trivalence condition changes the cohomology of the graph complexes only by a list of known classes, and the omission of graphs with tadpoles does not change the cohomology further. Notice also that graphs with multiple edges always have sign-reversing automorphism (flipping two parallel edges) and they vanish modulo defining relationships.

Although graph complexes are simple objects easy to define, their cohomology is still largely unknown [3].

Remark 1. *"The finite type invariants" in Knot theory can be recast in terms of "hairy graphs" (ordinary graphs with external legs ("hairs")). When all vertices are trivalent, this gives rise to the Vassiliev invariants* [2][7].

Remark 2. *Graph complexes give special classes in diffeomorphism groups of spheres. Maxim Kontsevich's characteristic classes are invariants of framed smooth fiber bundles with homology sphere fibers* [2][8].

Remark 3. *Lie decorated graph complexes describe the cohomology of the automorphisms of a free group* [2][9].

2 Generalization

The complex (GC, d^*) suggests a generalization, where vertex splitting (i.e. insertion of the complete graph on two vertices) is replaced by insertion of **the complete graph** on $k \geq 1$ vertices (a complete graph is a graph in which each pair of graph vertices is connected by an edge; the case $k = 1$ means adding a loop (an edge that connects a vertex to itself); the case $k = \infty$ is also meaningful). Let m_k be the corresponding map.

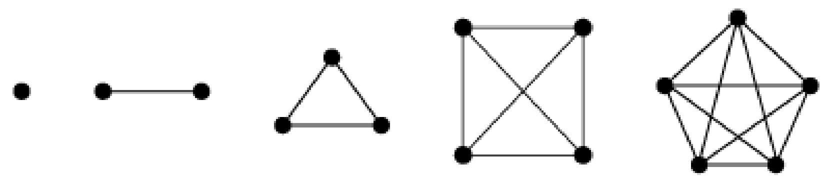

Figure 5: Complete graphs on n vertices, for n between 1 and 5

Remark 4. *The blowup of a graph is obtained by replacing every vertex with a finite collection of copies so that the copies of two vertices are adjacent if and only if the originals are. If every vertex is replaced with the same number of copies, then the resulting graph is called a balanced blowup. It comes from* ***complete bipartite graphs*** *(a graph whose vertices can be partitioned into two subsets such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph).*

Proposition 1. *For any $i, j \geq 2$*

$$[m_i, m_j] = (j - i) \cdot m_{i+j}.$$

Proof. From the definition of m_k and the properties of a complete graph (i.e. any permutation is possible) it becomes noticeable. Note that the complete graph with k vertices has $\frac{k(k-1)}{2}$ (the triangular numbers) undirected edges.▲

Remark 5. ***Witt algebra****?*
Note that m_1, m_2, m_3, m_4 generate everything.

Remark 6. *Contracting an edge of a graph does not change the Euler characteristic as both the vertex number and the edge number decrease by one.*

Similar to an edge contraction, let's use a **vertex contraction**:

$$\delta(G, o) = \sum_{v \in V(G)} (G/v, o_{G/v}),$$

where G/v is the graph obtained from G by deleting the vertex v (a vertex-deleted subgraph is a subgraph formed by deleting exactly one vertex) and $o_{G/v}$ is the induced orientation.

Remark 7. ***Reconstruction Conjecture*** *in graph theory asks: are graphs uniquely determined by their subgraphs? It has been shown by Béla Bollobás* [10] *that almost all graphs are reconstructible, i.e. the probability that a randomly chosen graph of n vertices is not reconstructible goes to 0 as n tends to infinity. Check also the paper* [11] *of Richard Stanley about switching-reconstructible graphs (for a vertex $x \in V(G)$ the graph G_x, obtained from G*

by deleting all edges incidents to x and adding edges joining x to every vertex not adjacent to x in G, is called a vertex-switching, i.e. to switch a vertex of a graph is to exchange its sets of neighbours and non-neighbours; Richard Stanley proved that a graph on n vertices is switching-reconstructible if $n \not\equiv 0 \mod 4$).

There is a stronger version of the conjecture. Set Reconstruction Conjecture: any two graphs on at least four vertices with the same sets of vertex-deleted subgraphs are isomorphic.

<u>*If Set Reconstruction Conjecture is true, then δ has inverse.*</u>

Reconstruction Conjecture is true if all 2-connected graphs (a connected graph is called 2-connected, if the induced vertex-deleted subgraph for every vertex is connected) are reconstructible. The conjecture has been verified for a number of infinite classes of graphs, for example: regular graphs (a regular graph is a graph where each vertex has the same number of neighbors), trees, circle graphs (a circle graph is the intersection graph of a chord diagram, i.e. it is an undirected graph whose vertices can be associated with a finite system of chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other), outerplanar graphs (a finite graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or of the complete bipartite graph $K_{2,3}$), maximal planar graphs (all faces (including the outer one) are then bounded by three edges).

Proposition 2. *For any $k > 2$*

$$[\delta, m_k] = k \cdot m_{k-1}.$$

Moreover, we have

$$[\delta, m_1] = \delta,$$

$$[\delta, m_2] = 2 \cdot cardinality \cdot id,$$

where the cardinality is the cardinality of the vertex set.

Proof. Rearranging terms in $\delta \circ m_k$.▲

Remark 8. *The notion of a* ***strong homotopy Lie (or L_∞) algebra*** *is well-known in algebraic homotopy theory, where it originated. It is obtained by allowing for a countable family of multilinear antisymmetric operations of all arities $n \geq 1$, constrained by a countable series of generalizations of the Jacobi identity known as the L_∞ identities. This notion admits specializations indexed by subsets $S \subseteq \mathbb{N}$ of arities and which are defined by requiring vanishing of all products of arities not belonging to S. This leads to the notion of L_S algebra. The case $S = \{n\}$, when only a single product of arity n is non-vanishing, recovers the notion of n-Lie algebras, see* [12] *for more details.*

Remark 9. *Let m_k^l denote the corresponding map, where vertex splitting is replaced by insertion of the complete graph on $k \geq 1$ vertices with l multiple edges. Hence, Proposition 2, using differential d from Subsection 1.2 (edge contraction), has an extrapolation.*

Proposition 3. *Let $\Delta_{k-2}^k = [\delta, [\ldots, [\delta, m_k] \ldots]]$, i.e. $(k-2)$ times, for any $k > 2$, then for any $l \geq 0$*

$$\Delta_{k-2}^k \circ \Delta_{k+l-2}^{k+l} = 0.$$

Proof. It follows from $[\delta, m_k] = k \cdot m_{k-1}$ in Proposition 2. Note that $m_2 \circ m_2 = 0$ [1 − 2].▲

Remark 10. *There is a connection with the study of multiple zeta values and the Kashiwara-Vergne conjecture* [13][14][15].

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