Null Controllability of Nonlocal Hilfer Fractional Stochastic Differential Equations Driven by Fractional Brownian Motion and Poisson Jumps

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Abstract

This manuscript investigates a class of nonlocal Hilfer fractional stochastic differential equations driven by fractional Brownian motion, which is a special case of a self-similar process, Hermite processes with stationary increments with long-range dependence. The Hermite process of order 1 is fractional Brownian motion and of order 2 is the Rosenblatt process. We establish new sufficient conditions of exact null controllability for such stochastic settings by using fractional calculus and fixed point theorem. The derived result in this article is new in the sense that it generalizes many of the existing results in the literature, more precisely for fractional Brownian motion and Poisson jumps case of Hilfer fractional stochastic settings. However to reveal the contemporary applicative feature of null controllability authors have validated illustration of stochastic partial differential equations.

Keywords: Hilfer fractional derivative, Stochastic differential system, Fractional Brownian motion, Poisson jumps, Null controllability.

2010 AMS Subject Classification: 26A33, 34A12, 34A08, 93B05.

1 Introduction

Over the past fewdecades, the theory and applications of the fractional differential equations enjoy considerable importance in the field of science, signal processing, and many other areas. The systematic presentation of the applications of fractional differential equations could be seen in the books [28, 29, 30, 31] and the references therein.

The deterministic system often fluctuates due to environmental noise. So, it is important and necessary for us to deal with stochastic differential equations. As a results of its widespread use, the stability result of stochastic system has received extensive attention. Hence it is appropriate to move from deterministic models to stochastic ones. Stochastic differential equations are important as they have many applications in many disciplines, including science and engineering, etc. For more details on stochastic differential equations readers may refer to the monographs [24, 32, 33] and the articles therein [34, 35].

The fractional Brownian motion is the usual candidate to model phenomena due to its self-similarity of increments and long-range dependence. This fractional Brownian $w^{\mathtt{H}}$ is the continuous centered Gaussian process with covariance function described by

$$R^{\mathtt{H}}(t,s) \ := \ \mathbf{E}\left[w^{\mathtt{H}}(t)w^{\mathtt{H}}(s)\right] = \frac{1}{2}(t^{2\mathtt{H}} + s^{2\mathtt{H}} - |t-s|^{2\mathtt{H}}).$$

The parameter $\mathtt{H}$ characterizes all the important properties of the process, when $\mathtt{H} < \frac{1}{2}$ the increments are negatively correlated and the correlation decays more slowly than quadratically; when $\mathtt{H} > \frac{1}{2}$, the increments are positively correlated and the correlation decays so slowly that they are not summable, a situation which is commonly known as the long memory property. Natural candidates are the Hermite processes, these non-Gaussian stochastic processes appear as limits are called Non-Central Limit theorem [26]. The fractional Brownian motion can be expresses as a Wiener integral with respect to the standard Wiener process, i.e. the integral of a deterministic kernel with respect to a standard Brownian motion, the Hermite process of order 1 is fractional Brownian motion and of order 2 is the Rosenblatt process.

On the other hand, controllability problems for different kinds of dynamical systems have been studied by several authors, see [2, 3, 5, 8, 24, 4] and references therein. A weaker condition than exact controllability is the property of being able to steer all points exactly to the origin. This has important connections with the concept of stabilizability. Many authors have investigated the null controllability of various kinds of dynamical systems [3, 6, 8, 11, 12, 7]. The problem of controllability of nonlinear stochastic or deterministic system has been discussed in [13, 15, 11, 12]. Recently, basic theory of differential equations involving Caputo and Riemann-Liouville fractional derivatives can be found in [16, 17, 22] and references therein. Beside Caputo and Riemann-Liouville fractional derivatives, there exists a new definition of fractional derivative introduced by Hilfer, which generalized the concept of Riemann-Liouville fractional derivative and has many application, for more details, see [20, 21, 23].

Motivated by the aforementioned research works, in this manuscript we derive the sufficient conditions for the null controllability of the following class of nonlocal Hilfer fractional stochastic differential equations driven by Rosenblatt process with Poisson jumps

$$\begin{aligned} D_{0+}^{\alpha,\beta} y(t) &= Ay(t) + B\mathbf{u}(t) + f(t, y(t))dt + g(t, y(t))\frac{dw^{\mathtt{H}}(t)}{dt} \\ &\qquad + \int_{\Lambda} h(t, y(t), \nu)\widetilde{N}(dt, d\nu), \quad t \in J = [0, a], \\ I_{0+}^{(1-\alpha)(1-\beta)} y(0) &+ q(y) = y_0. \end{aligned} \tag{1}$$

Here $D_{0+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $0 \le \alpha \le 1$, $\frac{1}{2} \le \beta \le 1$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $(S(t))_{t\ge 0}$, on a separable Hilbert space $\mathcal{X}$. Suppose $\{w^{\mathtt{H}}(t)\}_{t\ge 0}$ is a fractional Brownian motion with Hurst parameter $\mathtt{H} \in (\frac{1}{2}, 1)$ defined on $(\Omega, \Im, \{\Im_t\}_{t\ge 0}, \mathbb{P})$ with values in Hilbert space $\mathcal{Y}$; $f : J \times \mathcal{X} \to \mathcal{X}$, $g : J \times \mathcal{X} \to \mathcal{L}_2^0(\mathcal{Y}, \mathcal{X})$, $h : J \times \mathcal{X} \times \Lambda \to \mathcal{X}$ and $q : \mathscr{C}(J, \mathcal{X}) \to \mathcal{X}$ appropriate functions, where $\mathcal{L}_2^0(\mathcal{Y}, \mathcal{X})$ be the space of all Q-Hilbert Schmidt operators from $\mathcal{Y}$ into $\mathcal{X}$ and the control function $u(\cdot)$ is given in $\mathcal{L}^2(J, \mathcal{U})$, the Hilbert space of admissible control functions with $\mathcal{U}$ as a separable Hilbert space. The symbol B stands for a bounded linear operator from $\mathcal{U}$ into $\mathcal{X}$. Also, the initial condition y_0 is an $\Im_0$ measurable $\mathcal{X}$-valued stochastic process independent of the Rosenblatt process $Z_{\mathtt{H}}$ with finite second moment. In $\widetilde{N}(dt, d\nu) = N(dt, d\nu) - dt(\lambda d\nu)$ the Poisson measure $\widetilde{N}(dt, d\nu)$ denotes the Poisson counting measure.

To the best of authors knowledge, there are limited works by considering the controllability results of the following nonlocal Hilfer fractional stochastic differential equations driven by fractional Brownian motion with Poisson jumps. The contributions of this manuscript exist in the following aspects:

(1) New sufficient conditions of exact null controllability of nonlocal Hilfer fractional stochastic differential equations driven by fractional Brownian motion with Poisson jumps is formulated.

(2) Fractional calculus theory and fixed point theorem is effectively used to derive the sufficient conditions of exact null controllability for such stochastic system.

(3) An example is proved to illustrate the obtained theoretical results.

2 Preliminaries

In this section, we recollect basic concepts, definitions and lemmas which will be used in the sequel to obtain the main results.

Definition 2.1. *[28, 29] The Riemann-Liouville fractional integral operator of order $\beta > 0$ for a function f can be defined as*

$$I_{0+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds, \quad t > 0$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *[20] The Hilfer fractional derivative of type $0 \le \alpha \le 1$ and order $0 < \beta < 1$ is defined as*

$$D_{0+}^{\alpha,\beta} f(t) = I_{0+}^{\alpha(1-\beta)} \frac{d}{dt} I_{0+}^{(1-\alpha)(1-\beta)} f(t)$$

Let $(\Omega, \Im, \{\Im_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $(\Im_t)$, $t \in [0,a]$ and Let $\mathcal{X}$, $\mathcal{Y}$ be real separable Hilbert spaces and $\mathcal{L}(\mathcal{Y},\mathcal{X})$ denote the space of all bounded linear operator from $\mathcal{Y}$ into $\mathcal{X}$. Let $Q \in \mathcal{L}(\mathcal{Y},\mathcal{Y})$ be an operator defined by $Qe_n - \lambda_n e_n$ with finite trace $tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \ge 0$ $(n = 1, 2, ...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, ...)$ is a complete orthonormal basis in $\mathcal{Y}$.

We define the infiite dimensional fractional Brownian motion on $\mathcal{Y}$ with covariance Q as

$$w^{\mathtt{H}}(t) = w_Q^{\mathtt{H}}(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^{\mathtt{H}}(t)$$

where $\beta_n^{\mathtt{H}}$ are real, independent fractional Brownian motions.

In order to define Wiener integrals with respect to the Q-fractional Brownian motion, we introduce the space $\mathcal{L}_2^0 = \mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$ of all Q-Hilbert-Schmidt operators $\psi : \mathcal{Y} \to \mathcal{X}$. We recall that $\psi \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0(\mathcal{Y},\mathcal{X})} = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \psi e_n \right\|^2 < \infty$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $< \upsilon, \psi >_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} < \upsilon e_n, \psi e_n >$ is a separable Hilbert space. Let $\phi(s); s \in [0,a]$ be a function with values in $\mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$, the Wiener integral of ϕ with respect to $w^{\mathtt{H}}$ is defined by

$$\int_0^t \phi(s) dw^{\mathtt{H}}(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} \phi(s) e_n d\beta_n^{\mathtt{H}} = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} K^*(\phi e_n)(s) d\beta_n(s) \tag{2}$$

where β_n is the standard Brownian motion. Let $\mathscr{C}(J, \mathcal{L}_2(\Omega,\mathcal{X}))$ be the Banach space of all continuous maps from J into $\mathcal{L}_2(\Omega,\mathcal{X})$ satisfying $\sup_{0\le t\le \tau} \mathbf{E}\|y(t)\|^2 < \infty$.

Define $\mathbb{Y} = \{y : t^{(1-\alpha)(1-\beta)} y(t) \in \mathscr{C}(J, \mathcal{L}_2(\Omega,\mathcal{X}))\}$, with norm defined by

$$\|\cdot\|_{\mathbb{Y}} = \left(\sup_{t\in J} \mathbf{E} \left\| t^{(1-\alpha)(1-\beta)} y(t) \right\|^2 \right)^{\frac{1}{2}}.$$

Obviously, $\mathbb{Y}$ is a Banach space.

Lemma 2.1. *[26] If $\psi : [0,a] \to \mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$ satisfies $\int_0^a \|\psi(s)\|^2_{\mathcal{L}_2^0} < \infty$ then the above sum in () is well defined as $\mathcal{X}$-valued random variable and we have*

$$\mathbf{E} \left\| \int_0^t \psi(s) dw^{\mathtt{H}}(s) \right\|^2 \le 2\mathtt{H} t^{2\mathtt{H}-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

To study the exact null controllability of (1) we consider the fractional linear system

$$
\begin{aligned}
D_{0+}^{\alpha,\beta}y(t) &= Ay(t)+B\mathbf{u}(t)+f(t)+g(t)\frac{dw^{\mathtt{H}}(t)}{dt}, \quad t\in J=[0,a],\\
I_{0+}^{(1-\alpha)(1-\beta)}y(0) &= y_0,
\end{aligned}
\tag{3}
$$

associated with the system (1). Define

$$
\mathcal{L}_0^a\mathbf{u} = \int_0^a P_\beta(a-s)B\mathbf{u}(s)ds : \mathcal{L}_2(J,\mathcal{U})\to\mathcal{X},
$$

where $\mathcal{L}_0^a\mathbf{u}$ has a bounded inverse operator $(\mathcal{L}_0)^{-1}$ with values in $\mathcal{L}_2(J,\mathcal{U}/ker(\mathcal{L}_0^a))$, and

$$
\mathcal{N}_0^a(y,f,g) = S_{\alpha,\beta}(a)y+\int_0^a P_\beta(a-s)f(s)ds+\int_0^a P_\beta(a-s)g(s)dw^{\mathtt{H}}(s) : \mathcal{X}\times\mathcal{L}_2(J,\mathcal{U})\to\mathcal{X}.
$$

Definition 2.3. *The system (3) is said to be exactly null controllable on J if $Im\mathcal{L}_0^a\supset Im\mathcal{N}_0^a$.*

The system (3) is exactly null controllable if there exists $\gamma>0$ such that

$$
\|(\mathcal{L}_0^a)^*y\|^2 \geq \gamma\,\|(\mathcal{N}_0^a)^*y\|^2 \quad y\in\mathcal{X}.
$$

Lemma 2.2. *[7] Suppose that the linear system (3) is exactly null controllable on J. Then the linear operator $\mathcal{W}=(\mathcal{L}_0)^{-1}\mathcal{N}_0^a:\mathcal{X}\times\mathcal{L}_2(J,\mathcal{U})\to\mathcal{L}_2(J,\mathcal{U})$ is bounded and the control*

$$
\begin{aligned}
\mathbf{u}(t) &= -(\mathcal{L}_0)^{-1}\big[S_{\alpha,\beta}(a)y_0+\int_0^a P_\beta(a-s)f(s)ds+\int_0^a P_\beta(a-s)g(s)dw^{\mathtt{H}}(s)\big]\\
&= -\mathcal{W}(y_0,f,g)
\end{aligned}
$$

transfers the system (3) from y_0 to 0, where $\mathcal{L}_0$ is the restriction of $\mathcal{L}_0^a$ to $[ker\mathcal{L}_0^a]$

Definition 2.4. *If $y\in\mathscr{C}(J,\mathcal{L}_2(\Omega,\mathcal{X}))$ is a mild solution of (1) ifit satisfies*

$$
\begin{aligned}
y(t) &= S_{\alpha,\beta}(t)[y_0-q(0)]+\int_0^t P_\beta(t-s)\left[f(s,y(s))+B\mathbf{u}(s)\right]ds+\int_0^t P_\beta(t-s)g(s,y(s))dw^{\mathtt{H}}(s)\\
&+ \int_\Lambda\int_0^t P_\beta(t-s)g(r,y(r),\nu)\widetilde{N}(ds,d\nu), \quad t\in J.
\end{aligned}
$$

where

$$
\begin{aligned}
S_{\alpha,\beta}(t) &= I_{0+}^{\alpha(1-\beta)}P_\beta(t),\\
P_\beta(t) &= t^{\beta-1}T_\beta(t),\\
T_\beta(t) &= \int_0^\infty \beta\theta\psi_\beta(\theta)T(t^\beta\theta)d\theta,
\end{aligned}
$$

here

$$
\psi_\beta(\theta) = \sum_{n=1}^{\infty}\frac{(-\theta^{n-1})}{(n-1)!\Gamma(1-n\beta)sin(n\pi\alpha)} \quad \theta\in(0,\infty),
$$

is a function of Wright-type defined on $(0,\infty)$ and

$$
\int_0^\infty \theta^\zeta\psi_\beta(\theta)d\theta = \frac{\Gamma(1+\zeta)}{\Gamma(1+\beta\zeta)}, \quad \zeta\in(-1,\infty)
$$

and $\|T(t)\|\leq M$.

Lemma 2.3. *[27] The properties of the operators $S_{\alpha,\beta}$ and P_β are given by* *(i)* $\{P_\beta(t) : t > 0\}$ *is continuous in the uniform operator topology.*

(ii) *For any fixed* $t > 0$, $S_{\alpha,\beta}$ *and* P_β *are linear and bounded operators, and*

$$\begin{aligned} \|P_\beta(t)x\| &\leq \frac{Mt^{2(\beta-1)}}{\Gamma(\beta)^2}\|x\| \\ \|S_{\alpha,\beta}(t)x\| &\leq \frac{Mt^{2(\alpha-1)(\beta-1)}}{\Gamma(\alpha(1-\beta)+\beta)^2}\|x\|. \end{aligned}$$

(iii) $\{P_\beta(t) : t > 0\}$ *and* $\{S_{\alpha,\beta}(t) : t > 0\}$ *are strongly continuous.*

3 Exact null controllability

In this section, we formulate sufficient conditions for exact null controllability for the system (1) Before starting and proving our main results, we introduce the following hypotheses:

(H1) The linear system (3) is exactly null controllable on J.

(H2) The function $f : J \times \mathcal{X} \to \mathcal{X}$ is locally Lipschitz continuous, for all $t \in J$, $y, y_1, y_2 \in \mathcal{X}$, there exist constant $c_1 > 0$ such that

$$\begin{aligned} \|f(t, y_1) - f(t, y_2)\|^2 &\leq c_1 \|y_1 - y_2\|^2 \\ \|f(t, y)\|^2 &\leq c_1(1 + \|y\|^2). \end{aligned}$$

(H3) The function $g : J \times \mathcal{X} \to \mathcal{L}_2^0(\mathcal{Y}, \mathcal{X})$ is locally Lipschitz continuous, for all $t \in J$, $y, y_1, y_2 \in \mathcal{X}$, there exist constant $c_2 > 0$ such that

$$\begin{aligned} \|g(t, y_1) - g(t, y_2)\|^2 &\leq c_2 \|y_1 - y_2\|^2 \\ \|g(t, y)\|^2 &\leq c_2(1 + \|y\|^2). \end{aligned}$$

(H4) The function $h : J \times \mathcal{X} \times \Lambda \to \mathcal{X}$ is locally Lipschitz continuous, for all $t \in J$, $y, y_1, y_2 \in \mathcal{X}$, there exist constant $c_3 > 0$ such that

$$\begin{aligned} \int_\Lambda \|h(t, y_1, \nu) - f(t, y_2, \nu)\|^2 &\leq c_3 \|y_1 - y_2\|^2 \\ \int_\Lambda \|h(t, y, \nu)\|^2 &\leq c_3(1 + \|y\|^2). \end{aligned}$$

(H5) The function $q : \mathscr{C}(J, \mathcal{H}) \to \mathcal{X}$ is continuous, for all $t \in J$, $y, y_1, y_2 \in \mathscr{C}(J, \mathcal{X})$, there exist constant $c_4 > 0$ such that

$$\begin{aligned} \|q(y_1) - q(y_2)\|^2 &\leq c_4 \|y_1 - y_2\|^2 \\ \|q(y)\|^2 &\leq c_4(1 + \|y\|^2). \end{aligned}$$

Set $\Xi_1 = \frac{5M^2 c_4}{\Gamma^2(\alpha(1-\beta)+\beta)} + \frac{5M^2 a^{1+2\alpha(\beta-1)}}{(2\beta-1)\Gamma^2(\beta)}(c_1 + c_2 2\mathtt{H}T^{2\mathtt{H}-1} + c_3)$ and $\Xi_2 = 1 + \frac{5M^2\|B\|^2\|\mathcal{W}\|^2 a^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}$.

Theorem 3.1. *Assume that* (**H1**) – (**H5**) *hold, then the system (1) is exactly null controllable on* J*, provided that*

$$\Xi = \Xi_1 \Xi_2 < 1. \tag{4}$$

Proof. Define the operator $\mathcal{G}$ on $\mathbb{Y}$ as follows:

$$
\begin{aligned}
(\mathcal{G}y)(t) &= S_{\alpha,\beta}(t)\left[y_0 - q(x)\right] + \int_0^t P_\beta(t-s)\left[f(s,y(s)) + B\mathbf{u}(s)\right]ds \\
&+ \int_0^t P_\beta(t-s)g(s,y(s))dw^{\mathtt{H}}(s) + \int_0^t\int_\Lambda P_\beta(t-s)h(s,y(s),\nu)\widetilde{N}(ds,d\nu), \quad t\in J, \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{u}(t) &= \mathcal{W}[y_0 - q(y), f, g, h](t) \\
&= -(\mathcal{L}_0)^{-1}\Big\{S_{\alpha,\beta}(t)\left[y_0 - q(x)\right] + \int_0^t P_\beta(t-s)\left[f(s,y(s))\right]ds \\
&+ \int_0^t P_\beta(t-s)g(s,y(s))dw^{\mathtt{H}}(s) + \int_0^t\int_\Lambda P_\beta(t-s)h(s,y(s),\nu)\widetilde{N}(ds,d\nu)\Big\}
\end{aligned}
$$

It will be prove that $\mathcal{G}$ on $\mathbb{Y}$ into itself has a fixed point.
Step 1: The control function $\mathbf{u}(\cdot)$ is bounded on $\mathbb{Y}$ Now,

$$
\begin{aligned}
\|\mathbf{u}\|^2 &= \sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\|\mathbf{u}\|^2 \\
&\leq \|\mathcal{W}\|^2\Bigg[\frac{M^2}{\Gamma^2(\alpha(1-\beta)+\beta)}\Big\{\mathbf{E}\|y_0\|^2 + c_4(1+\mathbf{E}\|y\|^2)\Big\} + \frac{M^2a^{1+2\alpha(\beta-1)}}{(2\beta-1)\Gamma^2(\beta)}(1+\mathbf{E}\|y\|^2) \\
&\quad (c_1 + c_2 2\mathtt{H}T^{2\mathtt{H}-1} + c_3)\Bigg]
\end{aligned}
$$

Step 2: $\mathcal{G}$ maps $\mathbb{Y}$ into itself. for $t\in J$, we have

$$
\begin{aligned}
\|(\mathcal{G}y)(t)\|^2_{\mathbb{Y}} &= \sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\|(\mathcal{G}y)(t)\|^2 \\
&\leq 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\Bigg[\mathbf{E}\left\|S_{\alpha,\beta}(t)\left[y_0 - q(x)\right]\right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\left\|\int_0^t P_\beta(t-s)\left[f(s,y(s))\right]ds\right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\left\|\int_0^t P_\beta(t-s)B\mathbf{u}(s)ds\right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\left\|\int_0^t P_\beta(t-s)g(s,y(s))dw^{\mathtt{H}}(s)\right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\left\|\int_0^t P_\beta(t-s)\int_\Lambda h(s,y(s),\nu)\widetilde{N}(ds,d\nu)\right\|^2\Bigg] \\
&\leq \Bigg[\frac{5M^2}{\Gamma^2(\alpha(1-\beta)+\beta)}\Big\{\mathbf{E}\|y_0\|^2 + c_4(1+\mathbf{E}\|y\|^2)\Big\} \\
&+ \frac{5M^2a^{1+2\alpha(\beta-1)}}{(2\beta-1)\Gamma^2(\beta)}(1+\mathbf{E}\|y\|^2)(c_1 + c_2 2\mathtt{H}T^{2\mathtt{H}-1} + c_3)\Bigg] \\
&\times\left[1 + \frac{5M^2\|B\|^2\|\mathcal{W}\|^2a^{2\beta-1}}{(2\beta-1)\Gamma^2(\beta)}\right] < \infty.
\end{aligned}
$$

Thus $\mathcal{G}$ maps $\mathbb{Y}$ into itself.
Step 3: $\mathcal{G}$ is continuous on J.

Let $0 < t \leq a$ and $\zeta > 0$ be sufficiently small, then

$$\begin{aligned}
&\|(\mathcal{G}y)(t+\zeta)-(\mathcal{G}y)(t)\|_{\mathbb{Y}}^2 \\
&= \sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\,\|(\mathcal{G}y)(t+\zeta)-(\mathcal{G}y)(t)\|^2 \\
&\leq 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)} \times \mathbf{E}\,\|S_{\alpha,\beta}(t+\zeta)-S_{\alpha,\beta}(t)\,[y_0-q(x)]\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)} \times \mathbf{E}\left\|\int_0^{t+\zeta} P_\beta(t+\zeta-s)B\mathbf{u}(s)ds - \int_0^t P_\beta(t-s)B\mathbf{u}(s)ds\right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)} \times \mathbf{E}\left\|\int_0^{t+\zeta} P_\beta(t+\zeta-s)\,[f(s,y(s))]\,ds - \int_0^t P_\beta(t-s)\,[f(s,y(s))]\,ds\right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)} \times \mathbf{E}\Bigg\|\int_0^{t+\zeta} P_\beta(t+\zeta-s)\,[g(s,y(s))]\,dw^{\mathtt{H}}(s) \\
&\qquad\qquad - \int_0^t P_\beta(t-s)\,[g(s,y(s))]\,dw^{\mathtt{H}}(s)\Bigg\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)} \times \mathbf{E}\Bigg\|\int_0^{t+\zeta}\int_\Lambda P_\beta(t+\zeta-s)\,[h(s,y(s),\nu)]\,\widetilde{N}(ds,d\nu) \\
&\qquad\qquad - \int_0^t\int_\Lambda P_\beta(t-s)\,[h(s,y(s),\nu)]\,\widetilde{N}(ds,d\nu)\Bigg\|^2
\end{aligned} \tag{6}$$

By using Lemma 2.3, (**H2**) – (**H5**), the right hand side of (6) tends to zero as $\zeta \to 0$. Hence $\mathcal{G}$ is continuous on J.

Step 4: $\mathcal{G}$ is contraction on $\mathbb{Y}$.

For any $t \in J$ and $y_1, y_2 \in \mathbb{Y}$, we have

$$\begin{aligned}
&\|(\mathcal{G}y_1)(t)-(\mathcal{G}y_2)(t)\|_{\mathbb{Y}}^2 \\
&= \sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\,\|(\mathcal{G}y_1)(t)-(\mathcal{G}y_2)(t)\|^2 \\
&\leq 5\sup_{t\in J} t^{2(1-\alpha)(1-\beta)}\mathbf{E}\Bigg[\,\|(S_{\alpha,\beta})(t)[q(y_1)-q(y_2)]\|^2 \\
&+ \left\|\int_0^t P_\beta(t-s)[B\mathcal{W}[y_0-q(y_1),f,g,h](s) - B\mathcal{W}[y_0-q(y_2),f,g,h](s)]ds\right\|^2 \\
&+ \left\|\int_0^t P_\beta(t-s)[f(s,y_1(s))-f(s,y_2(s))]ds\right\|^2 + \left\|\int_0^t P_\beta(t-s)[g(s,y_1(s))-g(s,y_2(s))]dw^{\mathtt{H}}(s)\right\|^2 \\
&+ \left\|\int_0^t P_\beta(t-s)\int_\Lambda [h(s,y_1(s),\nu)-h(s,y_2(s),\nu)]\widetilde{N}(ds,d\nu)\right\|^2\Bigg] \\
&\leq \Xi\mathbf{E}\,\|y_1-y_2\|^2.
\end{aligned}$$

Hence, $\mathcal{G}$ is a contraction in $\mathbb{Y}$ using (4). From the Banach fixed point theorem, $\mathcal{G}$ has a unique fixed point. Therefore the system (1) is exact null controllable on J.

□

4 An example

Consider the following Hilfer fractional stochastic partial differential system driven by Rosenblatt process with Poisson jumps

$$\begin{aligned} D_{0^+}^{\beta,\frac{2}{3}} y(t,\xi) &= \frac{\partial^2}{\partial \xi^2} y(t,\xi) + \mathbf{u}(t,\xi) + f(t,y(t,\xi)) + g(t,y(t,\xi))\frac{dw^{\mathtt{H}}(t)}{dt} \\ &+ \int_\Lambda h(t,y(t,\xi),\nu)\widetilde{N}(dt,d\nu), \quad t\in J, \quad 0<\xi<1, \\ y(t,0) &= y(t,1)=0, \quad t\in J, \\ I_{0^+}^{\frac{1}{3}(1-\alpha)}(y(0,\xi)) + \sum_{i=1}^n k_i y(t_i\xi) &= y_0(\xi), \quad 0\le \xi\le 1, \end{aligned} \tag{7}$$

where $D_{0^+}^{\beta,\frac{2}{3}}$ is a Hilfer fractional derivative of order $0 \le \alpha \le 1$, $\beta = \frac{2}{3}$, $I^{\frac{1}{3}(1-\alpha)}$ is the Riemann-Liouville integral of order $\frac{2}{3}(1-\alpha)$ and $\{w^{\mathtt{H}}(t)\}_{t\ge 0}$ is a fractional Brownian motion with Hurst parameter $\mathtt{H} \in (\frac{1}{2},1)$. Let $f(t,y(t,\xi)), g(t,y(t,\xi),\nu), h(t,y(t,\xi),\nu)$ are given functions. Also, Let $A : \mathcal{X} \to \mathcal{X}$ be defined by $Ax = \frac{\partial^2}{\partial\xi^2}x$ with domain $D(A) = \Big\{ x \in \mathcal{X} : x, \frac{\partial x}{\partial \xi}$ are absolutely continuous, and $\frac{\partial^2 x}{\partial \xi^2} \in \mathcal{X}, x(0) = x(1) = 0 \Big\}$. It is known that A is self-adjoin and has the eigenvalues $\lambda_n = -n^2\pi^2, n \in \mathbb{N}$, with the corresponding normalized eigen-vectors $e_n(\xi) = \sqrt{2} sin(n\pi\xi)$. Furthermore, A generates a strongly continuous semigroup of bounded linear operator $S(t), t \ge 0$, on a separable Hilbert space $\mathcal{X}$ which is given by

$$S(t)x = \sum_{n=1}^\infty (x_n,e_n)e_n = \sum_{n=1}^\infty 2e^{-n^2\pi^2 t} sin(n\pi\xi)\int_0^t sin(n\pi\xi)\xi(\varsigma)d\varsigma, \quad x\in\mathcal{X}.$$

Define the fractional Brownian motion in $\mathbb{Y}$ by

$$w^{\mathtt{H}}(t) = \sum_{n=1}^\infty \sqrt{\lambda_n \beta^{\mathtt{H}}(t)} e_n,$$

where $\mathtt{H} \in (\frac{1}{2},1)$ and $\{\beta_n^{\mathtt{H}}\}_{n\in\mathbb{N}}$ is a sequence of one-dimensional fractional Brownian motions mutually independent.

If $\mathbf{u} \in \mathcal{L}_2(J,\mathcal{X})$, then $B = I$, $B^* = I$. Now we consider

$$\begin{aligned} D_{0^+}^{\beta,\frac{2}{3}} x(t,\xi) &= \frac{\partial^2}{\partial \xi^2} x(t,\xi) + \mathbf{u}(t,\xi) + f(t,\xi)) + g(t,x(t,\xi))\frac{dw^{\mathtt{H}}(t)}{dt} \\ &+ \int_\Lambda h(t,y(t)\xi),\nu)\widetilde{N}(dt,d\nu), \quad t\in J, \quad 0<\xi<1, \\ y(t,0) &= y(t,1)=0, \quad t\in J, \\ I_{0^+}^{\frac{1}{3}(1-\alpha)}(y(0,\xi)) + \sum_{i=1}^n k_i y(t_i\xi) &= y_0(\xi), \quad 0\le \xi\le 1, \end{aligned} \tag{8}$$

The system (9) is exact null controllability if there is a $\gamma > 0$, such that

$$\int_0^a \left\|B^* P_\beta^*(a-s)x\right\|^2 \ge \gamma\left[\left\|S_{\alpha,\beta}^*(a)x\right\|^2 + \int_0^a \left\|P_\beta^*(a-s)x\right\|^2 ds\right].$$

or equivalently

$$\int_0^a \|P_\beta(a-s)x\|^2 \ge \gamma\left[\|S_{\alpha,\beta}(a)x\|^2 + \int_0^a \|P_\beta(a-s)x\|^2 ds\right].$$

If $f = g = h = 0$ in (8), then the fractional linear system is exactly null controllable if

$$\int_0^a \|P_\beta(a-s)x\|^2 \, ds \quad \geq \quad a \, \|S_{\alpha,\beta}(a)x\|^2 .$$

Thus,

$$\int_0^a \|P_\beta(a-s)x\|^2 \, ds \quad \geq \quad \frac{a}{1+a}\gamma \left[\|S_{\alpha,\beta}(a)x\|^2 + \int_0^a \|P_\beta(a-s)x\|^2 \, ds \right].$$

Hence, the linear system (8) is exactly null controllable on J.

Clearly, the functions $f : J \times \mathcal{X} \to \mathcal{X}$, $g : J \times \mathcal{X} \to \mathcal{L}_2^0(\mathcal{Y}, \mathcal{X})$, $h : J \times \mathcal{X} \times \Lambda \to \mathcal{X}$ and $q : \mathscr{C}(J, \mathcal{X}) \to \mathcal{X}$ follows:

$$\begin{aligned} f(t,x) &= f(t, y(t,\xi)), \\ g(t,x) &= g(t, y(t,\xi)), \\ h(t,x,\nu) &= h(t, y(t,\xi), \nu), \\ q(x) &= \sum_{i=1}^{n} k_i y(t_i, \xi), \end{aligned}$$

which satisfy (**H2**) $-$ (**H5**). By k_i, $i = 1, 2, ..., n$; M, c_1, c_2, c_3, c_4 are constant such that $\Xi < 1$. Hence, all the hypotheses of Theorem 3.1 are satisfied. Hence the system (8) is exact null controllable on $[0, a]$.

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