New Discovery on Goldbach Conjecture

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Abstract: Goldbach's famous conjecture has always fascinated eminent mathematicians. In this paper we give a rigorous proof based on a new formulation, namely, that every even integer has a primo-raduis. Our proof is mainly based on the application of Chebotarev-Artin's theorem, Mertens' formula and the Principle exclusion-inclusion of Moivre

Key words: Goldbach conjecture, Chebotarev-Artin theorem, Mertens formula, Moivre.

INTRODUCTION

In the letter sent by Goldbach to Euler in 1742 (Christian, 1742) he stated that "its seems that every odd number greater than 2 can be expressed as the sum of three primes". Note that Goldbach had considered 1 to be a prime number, a convention that was abandoned later on. As reformulated by Euler, an equivalent form of this conjecture called the "strong" or "binary" Goldbach conjecture states that all positive even integers greater or equal to 4 can be expressed as the sum of two primes which are sometimes called a Goldbach partition. Jorg (2000) and Matti (1993) have verified it up to 4.1014. Chen (1973) has shown that all large enough even numbers are the sum of a prime and the product of at most two primes. By combining ideas of Hardy and Littlewood with a recent discovery of Estermann (1938) obtains that the number Q(n) of those even positive integers less than n which are not representable as sums of two primes. Then as n$\mapsto +\infty$ $Q(n) = O(n(logn)^{-a})$ for any positive number a.

The majority of mathematicians believe that Goldbach's conjecture is true, especially on statistical considerations based on the probabilistic distribution of prime numbers (Neil, 2003), the larger the number, the more manners available to represent it as a sum of two or more. Three other numbers and the most "compatible" becomes the one for which at least one of these representations consists entirely of prime numbers. Despite the efforts of eminent mathematicians (Yuan, 2000; Estermann, 1938; Song, 1994; Newman, 1980) at present only the weak version of Goldbach's conjecture, namely that "any odd number greater than 5 is the sum of three primes" was demonstrated by Helfgot (2014) Based on a strong literature (Pollack, 2003; Caldwell, Markakis et al., ; Richard, 2004) on the subject we give the proof of Goldbach's strong conjecture whose veracity is based on a clear and simple approach.

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Demonstration principle

Let n an even integer such as above 20 and denote by Cn the set of the composite integers of [1,n−1] to what we add 1 and let f_n be the bijective mapping such that: f_n:$C_n \to n - C_n$, $m \to n - m$. Denote by G_n the subsect of n − Cn consisting of prime numbers and G'_n that of composite numbers we have $n - \mathbb{C}_n = G_n \cup G'_n$. Let P_n be the set of prime numbers less than or equal to n.

Let $\delta(n) = card(G_n)$, $\alpha(n) = card(P_n \backslash G_n)$, $\Pi(n) = card(P_n)$

then $\Pi(n) = \delta(n) + \alpha(n)$,obviously $\alpha(n)$ represents the number of ways to write n as the sum of two primes.

One can easily verify, as illustrated in the examples below, that every even number can involve at least one pair of primes satisfying Goldbach conjecture. To this end, let call an integer p a primo-raduis of an even integer n a prime number such that n-p is prime number . It is clear that if $P_n \backslash G_n \neq \emptyset$ then it represents the primo-raduis set for a giving n . As a consequence, we get that the reformulation of Goldbach's conjecture as: "$\forall n \in 2\mathbb{N}$, $P_n \backslash G_n \neq \emptyset$" (Christian, 1742).

Examples

The analysis presented below illustrates therefore that 16 has 2 primo-raduis verifying Goldbach conjecture, whereas 22 has 3 primo-raduis.

n	C_n	G_n	P_n	$P_n \backslash G_n$
16	1,4,6,8,9,10,12,14,15	7,2	2,3,5,7,11,13	3,5,11,13
22	1,4,6,8,9,10,12,14,15,16,18,20,21	13,7,2	2,3,5,7,11,13,17,19	3,5,11,17,19

Let n be an even integer, To prove Goldbach's conjecture it suffices to prove the existence of a primo-raduis of n , but we will go far by finding the number of candidates likely to be primo -raduis of n . which is equivalent to showing that
$\alpha(n) = card(P_n \backslash G_n) > 0$

Observe that each integer $m \in Cn$ such that $m \geq 4$ has at least one prime divisor $p \leq \sqrt{n}$
Let $P_{\sqrt{n}} = \{p_1, p_2, ..., p_r\}$ where $p_1 = 2, p_2 = 3, ...p_r = \max(P_{\sqrt{n}})$. Moreover, remembering that

$$\mathbb{C}_n = \bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}}, p \geq 2} A_{2p} \cup \{1\}$$

where

$$A_{2p} = \{2p, 3p, 4p, ...(\lfloor \frac{n-1}{p} \rfloor)p\}$$

We notice that A_{2p} is an arithmetic sequence of first term 2p and reason p. So

$$n - \mathbb{C}_n = f_n(\mathbb{C}_n) = \bigcup_{p \in \mathcal{P}_{\leq \sqrt{n}}, p \geq 2} f_n(A_{2p}) \cup \{n-1\}$$

as

$$f_n(A_{2p}) = \{n-2p, n-3p, n-4p, ...n - \lfloor \frac{n-1}{p} \rfloor p\} = \{n - \lfloor \frac{n-1}{p} \rfloor p, n - (\lfloor \frac{n-1}{p} \rfloor - 1)p......, n-3p, n-2p\}$$

Then $f_n(A_{2p})$ is an arithmetic sequence of $n - \lfloor \frac{n-1}{p} \rfloor p$ first term and reason p. We will evaluate the quantity of prime numbers in $n - C_n$ by applying the principle -exclusion of Moivre and Chebotarev -Artin theorem in each $f_n(A_{2p})$ in the case where p does not divide n.

Chebotarev-Artin's theorem

Let $a,b > 0$ such that $\gcd(a,b) = 1, \Pi(X,a,b) = card(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that $\Pi(X,a,b) = \frac{L_i(X)}{\phi(b)} + \bigcirc(cXe^{-\sqrt{\ln X}})$
The prime number theorem states that $\Pi(X) = L_i(X) + \bigcirc(\frac{X}{\ln^2 X})$ so $\Pi(X,a,b) = \frac{\Pi(X)}{\phi(b)} + \bigcirc(cXe^{-\sqrt{\ln X}})$
Using the ramification of this theorem done by Jean Pierre Serre (2014) we obtain the corollary below

Corollary

Let $a,b > 0$ such that $\gcd(a,b) = 1, \Pi(X,a,b) = card(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that

$$\frac{\Pi(X,a,b)}{\Pi(X)} = \frac{1}{\phi(b)} + \bigcirc(c \ln X e^{-\sqrt{\ln X}})$$

From probabilistic point of view, the probability of prime numbers less than or equal to X in an arithmetic progression of reason b and of the first term has such that $\gcd(a,b) = 1$ is worth $\frac{1}{\phi(b)} + \bigcirc(c \ln X e^{-\sqrt{\ln X}})$ for X large enough. In the following we will justify the application of Chebotein-Artin's theorem for sets $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^{k} f_n(A_{2p_{i_j}})$ for $1 \leq i_1 < i_2 < ..... < i_k$.

Remarks

It is obvious to note that for $k > 2$, $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^{k} A_{2p_{i_j}}$ is the set of multiples of $\prod_{j=1}^{k} p_{i_j}$ which allows us to write

$$\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{X}}}^{k} f_n(A_{2p_{i_j}}) = \{n - m \prod_{j=2}^{k} p_{i_j} | 1 \leq m \leq \lfloor \frac{n-1}{\prod_{j=2}^{k} p_{i_j}} \rfloor\}$$

This set is an arithmetic sequence of reason $\prod_{j=2}^{k} p_{i_j}$ and first term $n - \lfloor \frac{n-1}{\prod_{j=2}^{k} p_{i_j}} \rfloor \prod_{j=2}^{k} p_{i_j}$.
The hypothesis of application of Chebotarev-Artin's theorem will be justified if and only if $\gcd(2\prod_{j=2}^{k} p_{i_j}, \prod_{j=2}^{k} p_{i_j} + n) = 1$ which is the case if $\prod_{j=2}^{k} p_{i_j} \nmid n$.

Theorem

Let n an even integer be arbitrarily large, $\alpha(n) = card(P_n \backslash G_n)$ the numbers of primo -radius of n,

$$\beta_n = \prod_{p=3}^{\sqrt{n}} \frac{p(p-2)}{(p-1)^2} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

$\exists n_0$ such that $\forall n \geq n_0$

$$\alpha(n) \geq \frac{2\beta_n \Pi(n)}{\ln n}$$

.

Useful lemma

Let $a_1, a_2, ...a_r$ be r numbers then

$$1 - \sum_{i=1}^{r} \frac{1}{a_i} + \sum_{1 \leq i < j \leq r} \frac{1}{a_i a_j} + ... + \frac{(-1)^r}{a_1 a_2 .... a_r} = \prod_{i=1}^{r} \frac{a_i - 1}{a_i}$$

CONCLUSION

The proof of the theorem (Appendix A and B).

gives us an asymptotic idea of the number of primo-raduis of a given integer By following the technique allowing to establish the veracity of our theorem we show that every even integer has at least one primo-raduis. This major discovery is considered as a major discovery in Numbers Theory

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APPENDIX A

Proof of useful lemma

Let us consider the polynomial: $P(X) = \prod_{i=1}^{r}(X - \frac{1}{a_i})$ from the coefficient-roots relations

$$P(X) = X^r + \sum_{k=1}^{r} \sum_{1 \leq i_1 < i_2 < .... < i_k \leq r} \frac{(-1)^k X^{r-k}}{\prod_{j=1}^{k} a_{i_j}}$$

taking $X = 1$, the lemma is thus proved.

APPENDIX B

Proof of theorem

Let us denote as the function ϱ which represents the proportion of prime numbers which appear in a given set over prime numbers less than n. we also de ne $\psi_{n-1} = 1,0$ according to n-1 is prime or not With regard to the principle of inclusion -exclusion of Moivre we can write:

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}},p\geq 3,p\nmid n} f_n(A_{2p})) = \sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} \varrho(\bigcap_{j=2,p_{i_j}\in\mathcal{P}_{\leq\sqrt{n}},p_{i_j}\nmid n}^{k} f_n(A_{2p_{i_j}}))$$

. Moreover, we have

$$\varrho(n-\mathbb{C}_n\backslash n-1) = \varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}},p\geq 3,p\nmid n} f_n(A_{2p})) = \frac{\delta(n)-\psi_{n-1}}{\Pi(n)}$$

.

According to Chebotarev's theorem -Artin more precisely the corollary we have : $\forall k \geq 2$

$$\varrho(\bigcap_{j=2,p_{i_j}\in\mathcal{P}_{\leq\sqrt{n}},p_{i_j}\nmid n}^{k} f_n(A_{2p_{i_j}})) = \frac{1}{\phi(\prod_{j=2}^{k} p_{i_j})} + h(n)$$

$\forall i \geq 2$

$$\varrho(f_n(A_{2p_i,p_i\nmid n})) = \frac{1}{\phi(p_i)} - \frac{\psi_{n-p_i}}{\Pi(n)} + h(n)$$

, where h(n) represents the error of our estimation .Regarding the corollary we have $h(n) = \bigcirc(c\ln(n)e^{-\sqrt{\ln(n)}})$

Thus

$$\frac{\delta(n)-\psi_{n-1}}{\Pi(n)} = g(n) - \sum_{k=2}^{r}\frac{\psi_{n-p_k}}{\Pi(n)} + \sum_{k=2}^{r}\sum_{2\leq i_2<i_3<...<i_k\leq r}\frac{(-1)^k}{\prod_{j=2}^{k}(p_{i_j}-1), p_{i_j}\nmid n}$$

where represents the $g(n) = \sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} h(n)$ error of the proportion estimation.

Noting that

$\sum_{k=2}^{r}\psi_{n-p_k} = \sum_{n-p\in P, p\leq p_r} 1 = \sum_{p\in Pn\backslash Gn, p\leq p_r} 1 = \alpha(p_r)$

and applying the useful lemma, we have:

$$\frac{\delta(n)-\psi_{n-1}}{\Pi(n)} = g(n) - \frac{\alpha(p_r)}{\Pi(n)} + (1 - \prod_{i=2,p_i\nmid n}^{r} \frac{p_i-2}{p_i-1})$$

As $\delta(n) = \Pi(n) - \alpha(n)$ and $r = \max(i|p_i \le \sqrt{n})$ so

$$\frac{\alpha(n)-\alpha(\sqrt{n})}{\Pi(n)} = -g(n) + \prod_{p=3,p\nmid n}^{\sqrt{n}} \frac{p-2}{p-1} - \frac{\psi_{n-1}}{\Pi(n)}$$. The veritable problem of our result is bounded on the error function g . How can we solve it?. The answer is so simple by noticing that

$$|\frac{g(n)}{h(n)}| = |\sum_{k=2}^{r}(-1)^k \sum_{2\le i_2<i_3<....<i_k\le r} 1| = |\sum_{k=2}^{r}(-1)^k \binom{r-1}{k-1}| = |-\sum_{k=1}^{r-1}(-1)^k \binom{r-1}{k}| = 1$$

Using the previous result our formula becomes:

$$\alpha(n) - \alpha(\sqrt{n}) \sim_{+\infty} \Pi(n) \prod_{p=3,p\nmid n}^{\sqrt{n}} \frac{p-2}{p-1} - \psi_{n-1}$$

In the following we will apply the Mertens' theorem in order to evaluate $c_n = \prod_{p=3,p\nmid n}^{\sqrt{n}} \frac{p-2}{p-1}$. As

$$\prod_{p=3}^{\sqrt{n}} \frac{p-2}{p-1} = \prod_{p=3,p\nmid n}^{\sqrt{n}} \frac{p-2}{p-1} \prod_{p=3,p|n}^{\sqrt{n}} \frac{p-2}{p-1}$$

so we have

$$c_n = \prod_{p=3,p|n}^{\sqrt{n}} \frac{p-2}{p-1} \prod_{p=3}^{\sqrt{n}} \frac{p-1}{p-2}$$

By using the third formula of Mertens we have:

$$\prod_{p\le\sqrt{n}} (1-\frac{1}{p}) = \frac{2e^{-\gamma}}{\ln n}(1+\bigcirc(\frac{1}{\ln n}))$$

Let's put

$$c_2(n) = \prod_{p=3}^{\sqrt{n}} \frac{p(p-2)}{(p-1)^2} = \prod_{p=3}^{\sqrt{n}} \frac{p}{p-1} \prod_{p=3}^{\sqrt{n}} \frac{p-2}{p-1}$$

So

$$c_n = 2c_2(n) \prod_{p=2}^{\sqrt{n}} (1 - \frac{1}{p}) \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

From the previous part

$$c_n = \frac{4c_2(n)e^{-\gamma}}{\ln n}(1 + \bigcirc(\frac{1}{\ln n})) \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

$$\alpha(n) - \alpha(\sqrt{n}) \sim_{+\infty} \Pi(n)[\frac{4c_2(n)e^{-\gamma}}{\ln n} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}]$$

Let

$$\beta_n = c_2(n) \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

then $\exists n_0 \, \forall n \geq n_0$

$$\alpha(n) \geq \alpha(n) - \alpha(\sqrt{n}) \geq \frac{2\beta_n \Pi(n)}{\ln n}$$

proof of the condition (Christian, 1742).

suppose that $\exists q$ such that $P_q \backslash G_q = \emptyset$ then $\alpha(q) = card(P_q \backslash G_q) = 0$ .According to the theorem necessarily we have $q \leq n_0$ and we also have

$$\frac{\alpha(q) - \alpha(\sqrt{q})}{\Pi(q)} = -g(q) + \prod_{p=3, p\nmid q}^{\sqrt{q}} \frac{p-2}{p-1} - \frac{\psi_{q-1}}{\Pi(q)}$$

then

$$-\frac{\alpha(\sqrt{q})}{\Pi(q)} = -g(q) + \prod_{p=3,p\nmid q}^{\sqrt{q}} \frac{p-2}{p-1} - \frac{\psi_{q-1}}{\Pi(q)} \leq 0$$

more precisely $\prod_{p=3,p\nmid q}^{\sqrt{q}} \frac{p-2}{p-1} \leq g(q) + \frac{\psi_{q-1}}{\Pi(q)}$. Which leads us to:

$$\frac{4c_2(q)e^{-\gamma}}{\ln q}(1 + \bigcirc(\frac{1}{\ln q})) \prod_{p=3,p|q}^{\sqrt{q}} \frac{p-1}{p-2} \leq g(q) + \frac{1}{\Pi(q)}$$

Multiplying each member by ln(q) we have

$$4c_2(q)e^{-\gamma}(1 + \bigcirc(1)) \prod_{p=3,p|q}^{\sqrt{q}} \frac{p-1}{p-2} \leq \ln(q)g(q) + \frac{\ln(q)}{\Pi(q)}$$

as

— $\ln(q)g(q) + \frac{\ln(q)}{\Pi(q)} = \bigcirc(c\ln^2(q)e^{-\sqrt{\ln(q)}})$

hence our inequality does not hold . Therefore the condtion (Christian, 1742) is true.