Random Walk and Reserves Modeling in Studying Pensions Funds Sustainability

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Abstract
Random walk is a stochastic process classic example, used to study a set of phenomena and, particularly, as in this article, models of reserves evolution. Random walks also allow the construction of significant complex systems and are also used as an instrument of analysis, being used in the sense of giving a theoretical characteristic to other types of systems. Our goal is primarily to study reserves to see how to ensure that pension funds are sustainable. This classic approach to the pension funds study makes it possible to draw interesting conclusions about the problem of reserves.

Keywords: *Reserves, Ruin, Random walks, Pensions funds*

JEL Classification: C18
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1 INTRODUCTION

Gambler's ruin suggests that growth events occur randomly, depending the survival on the stock of stored resources. In Gambler's ruin problem, reserves behave according to a simple random walk. This problem has been often presented in many works considering stochastic processes theory in contexts under the frameworks of Markov Chains, Random Walks, Martingales or even others. Billingsley (1986) or Feller (1968) solve problems in this topic using the classic first step analysis to obtain a difference equation, which approach we use in our study. In another basis, (Grimmett and Stirzaker, 1992) and (Karlin and Taylor, 1975) get resolutions through the Martingales Theory and apply it as an applications' example of the Martingales Stopping Time Theorem.
The paper is an updated approach of the work presented in SPMCS 2012, see (Filipe, Ferreira, and Andrade, 2012), and is organized as follows. In section 2. gambler's ruin problem is presented. In section 3. we approach random walks, modeling the general problem. Section 4. provides a ruin probability's particularization. In section 5. we approach the definition of pensions fund management policies to guarantee its sustainability. Section 6. concludes.
In Ferreira (2017, 2018), (Ferreira, Andrade, and Filipe, 2012) and (Andrade, Ferreira, and Filipe, 2012) more applications of stochastic processes can be seen in problems of reserves and pensions funds.

2 GAMBLER'S RUIN

We consider a gambler with an initial capital of x monetary units, intending to play a sequence of games till the gambler's wealth reaches k monetary units. We suppose that x and k are integer numbers satisfying the conditions $x > 0$ and $k > x$. In each game, there are two possible situations: winning 1 monetary unit with probability p or losing 1 monetary unit with probability $q = 1- p$. A question is posed in terms of knowing which will be the probability the gambler

gets in ruin before attaining his/her objective. This means that it is intended to know which the probability of losing the *x* monetary units is before adding gains of *k* – *x* monetary units.
Be $X_n, n = 1, 2, \ldots$ the outcome of the n^{th} game. Obviously, the variables $X_1, X_2, \ldots$ are *i.i.d.* random variables, which common probability function is: $P(X_n = 1) = p, P(X_n = -1) = q = 1 - p$.
Consequently, the gambler's wealth, his/her reserves after the n^{th} game represent the simple random walk as follows: $S_0 = x, S_n = S_{n-1} + X_n, n = 1, 2, \ldots$.
Aiming to get the gambler's ruin probability, let's consider this probability as $\rho_k(x)$. It relates to the probability that $S_n = 0$ and $0 < S_i < k, i = 0, 1, \ldots, n-1$ for $n = 1$ or $n = 2$ or ... . If $\rho_k(x)$ is conditioned to the result of the first game, and considering the Total Probability Law, we get the following:

$$\rho_k(x) = p\rho_k(x+1) + q\rho_k(x-1),\ 0 < x < k. \tag{1}$$

If $0 \leq x \leq k$, the difference equation presented is easily solved considering the border conditions

$$\rho_k(0) = 1,\ \rho_k(k) = 0 \tag{2}$$

getting:

$$\rho_k(x-1) = \begin{cases} \dfrac{1-\left(p/q\right)^{k-x}}{1-\left(p/q\right)^{k}}, p \neq \frac{1}{2} \\[2ex] \dfrac{k-x}{k},\ \ p = \frac{1}{2} \end{cases} \tag{3}$$

Be N_a the random walk $S_n$first passage time by *a*: $N_a = \min\{n \geq 0: S_n = a\}$. Now we can write $\rho_k(x) = P(N_0 < N_k | S_0 = x)$. It is appropriate to consider in (3) the limit as *k* converges to ∞ to evaluate $\rho(x)$, the ruin probability of a gambler infinitely ambitious. In this context, considering the simple random walk $S_n, \rho(x) = P(N_0 < \infty | S_0 = x)$, after
(3)

$$\rho(x) = \lim_{k\to\infty} \rho_k(x) = \begin{cases} (q/p)^x, if\ p > \frac{1}{2} \\[2ex] 1, \qquad \text{if}\, p \leq \frac{1}{2} \end{cases} \tag{4}$$

Being $\mu = E[X_n] = 2p - 1$, after (4), the ruin probability is 1 for the simple random walk at which the mean of the step is $\mu \leq 0 \Leftrightarrow p \leq \frac{1}{2}$.

3 THE GENERAL RANDOM WALK

In this section, we begin considering a fund in which contributions/pensions received/paid, per time unit, are described as a sequence of random variables $\xi_1, \xi_2, \ldots$ $(\eta_1, \eta_2, \ldots)$. Let's consider that $\xi_n(\eta_n)$ is the amount of the contributions/pensions that are received/paid by the fund during the n^{th} time unit and consequently $X_n = \xi_n - \eta_n$ is the reserves variation in the fund at the n^{th} time unit. If $X_1, X_2, \ldots$ is a sequence of non-degenerated *i.i.d* random variables, the stochastic process, representing the evolution of the fund reserves, since the value *x* till the amount $\tilde{S}_n$ after *n* time units, will be defined recursively as $\tilde{S}_0 = x, \tilde{S}_n = \tilde{S}_{n-1} + X_n,$ with $n = 1, 2, \ldots$. Such a process is a general random walk.
The aim is to study the fund ruin probability, i.e., the game reserves exhaustion probability.

We consider *x* and *k* real numbers fulfilling *x* > 0 and *k* > *x*. First, the evaluation of $\rho_k(x)$, the probability that the fund reserves decrease from an initial value *x* to a value in $(-\infty,0]$ before reaching a value in $[k,+\infty)$, is considered. Then, after the calculus of the limit, as seen in the previous section, the evaluation of $\rho(x)$, the eventual fund ruin probability is considered, by admitting in such a situation that the random walk - that represents its reserves - evolves with no restrictions at the right of 0.

This method that was presented here is known in the literature of stochastic processes as Wald's Approximation. The explanations by (Grimmett and Stirzaker,1992) and (Cox and Miller, 1965), relating this issue are diligently considered in our work. Here, we consider the $S_n = \tilde{S}_n - x$ process, i.e., the random walk $S_0 = 0, S_n = S_{n-1} + X_n, n = 1, 2, \ldots$, instead of $\tilde{S}_n$ process.

Accordingly, when we evaluate $\rho_k(x)$, it is the probability the process S_n is visiting the set $(-\infty,-x]$ before visiting the set $[k-x,+\infty)$ that is considered. And when we evaluate $\rho(x)$ it is only the probability that the process S_n goes down from the initial value 0 till a level lesser or equal than – *x* that is considered.

We begin considering now the non-null value θ for which the X_1 moments generator function assumes the value 1. It is assumed that such a θ exists, that is, θ satisfies

$$E\left[e^{\theta X_1}\right] = 1, \theta \neq 0 \qquad (5)$$

Defining the process as $M_n = e^{\theta S_n}, n = 0, 1, 2, \ldots$, it is now evident that $E[|M_n|] < \infty$ and also, after (5), $E[M_{n+1}|X_1, X_2, \ldots, X_n] = E\left[e^{\theta(S_n+X_{n+1})}|X_1, X_2, \ldots, X_n\right] = e^{\theta S_n}E\left[e^{\theta X_{n+1}}|X_1, X_2, \ldots, X_n\right] = M_n$. Consequently, the process M_n is a Martingale in what concerns the sequence of random variables $X_1, X_2, \ldots$. Consider now *N*, the S_n first passage time to outside the interval $(-x, k-x)$, $N = \min\{n \geq 0: S_n \leq -x \text{ or } S_n \geq k-x\}$.

The random variable *N* is a stopping time – or a Markov time – for which the following conditions are satisfied:

- $E[N] < \infty$,
- $E[|M_{n+1} - M_n||X_1, X_2, \ldots, X_n] \leq 2e^{|\theta|a}$, for $n < N$,
- $a = -x \vee a = k - x$.

We suggest giving a look at the work by (Grimmett and Stirzaker, 1992) on this issue. Under these conditions, we can resort to the Martingales Stopping Time Theorem and so:

$$E[M_n] = E[M_0] = 1 \qquad (6)$$

Besides,

$$E[M_n] = E\left[e^{\theta S_N}|S_N \leq -x\right]P(S_N \leq -x) + E\left[e^{\theta S_N}|S_N \geq k-x\right]P(S_N \geq kx) \qquad (7)$$

Realizing the approximations $E\left[e^{\theta S_N}|S_N \leq -x\right] \cong e^{-\theta x}$ and $E\left[e^{\theta S_N}|S_N \geq k-x\right] \cong e^{\theta(k-x)}$, and considering $P(S_N \leq -x) = \rho_k(x) = 1 - P(S_N \geq k-x)$, after (6) and (7), we obtain

$$\rho_k(x) \cong \frac{1-e^{\theta(k-x)}}{e^{-\theta x}-e^{\theta(k-x)}}, \text{when } E[X_1] \neq 0 \qquad (8)$$

This result is the Classic Approximation for the Ruin Probability in the conditions stated in (5). To admit a non-null solution θ for the equation $E\left[e^{\theta X_1}\right] = 1$ implies in fact to assume that $E[X_1] \neq 0$.

Going farer, we may consider a particular case, beyond the studied above, looking at the situation for which the equation $E\left[e^{\theta X_1}\right] = 1$ only solution is precisely $\theta = 0$; it means, the situation at which $E[X_1] = 0$. This case may be considered through the following passage to the limit:

$$\rho_k(x) \cong \lim_{\theta \to 0} \frac{1-e^{\theta(k-x)}}{e^{-\theta x}-e^{\theta(k-x)}} = \frac{k-x}{k}, \text{ when } E[X_1] = 0 \tag{9}$$

As for $\rho(x)$, the probability that the process S_n decreases eventually from the initial value 0 to a level lesser or equal than – *x*, is also got from (8), now for a different passage to the limit:

$$\rho(x) \cong \lim_{k \to \infty} \frac{1-e^{\theta(k-x)}}{e^{-\theta x}-e^{\theta(k-x)}} = e^{\theta x}, \text{ if } \theta < 0 \Leftrightarrow E[X_1] > 0 \tag{10}$$

Considering the previous section results on the simple random walk, it is effective to accept $\rho(x) = 1$ when $\theta \geq 0 \ \Leftrightarrow \mathrm{E}[\mathrm{X}_1] \leq 0$.

4 A RUIN PROBABILITY'S PARTICULARIZATION

Consider $X_1, X_2, \ldots$ is a sequence of independent random variables with normal distribution with mean μ and standard deviation σ. So, we can suppose X_n, is the fund reserves variation at the n^{th} time unit, normally distributed with those parameters. Now, the moments' generator function is: $E\left[e^{\theta X_1}\right] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{\theta x - \frac{(x-\mu)^2}{2\sigma^2}} dx = e^{\theta\mu + \frac{\theta^2\sigma^2}{2}}$. And equation (5) solution is $\theta = \frac{-2\mu}{\sigma^2}, \mu \neq 0$.

- So $\rho_k(x)$, the ruin probability, is acquired when substituting this result in (8), as follows:

$$\rho_k(x) \cong \frac{1-e^{-\frac{2\mu(k-x)}{\sigma^2}}}{e^{\frac{2\mu x}{\sigma^2}}-e^{-\frac{2\mu(k-x)}{\sigma^2}}}, \text{when } \mu \neq 0 \tag{11}$$

- It is obvious that this particularization does not influence the approximation to $\rho_k(x)$ when $\mu = 0$. As it was seen before, it is given by (9) and after (10), we have

$$\rho(x) \cong e^{-\frac{2\mu x}{\sigma^2}}, \text{ when } \mu > 0 \tag{12}$$

5 ASSETS AND LIABILITY MANAGEMENT POLITICS

It evidently follows from the matters presented so far that it is imperative to define pension fund management policies to guarantee its sustainability.

To do so, see for instance Ferreira (2017,2018), we assume that the assets value process of a pensions fund, may be represented by a geometric Brownian motion process $A(t) = be^{a+(\rho+\mu)t+\sigma B(t)}, \mu < 0, ab\rho + \mu\sigma > 0$, where $B(t)$ is a standard Brownian motion process. Suppose also that the fund liabilities value process performs such as the deterministic process $L(t) = be^{\rho t}$. Be also the stochastic process $Y(t) = ln\frac{A(t)}{L(t)} = a + \mu t + \sigma B(t)$: it is a generalized Brownian motion process, starting at a, with drift μ and diffusion coefficient

σ^2.Note also that the first passage time of the assets process $A(t)$ by the mobile barrier T_n, the liabilities process, is the first passage time of $Y(t)$ by 0, with finite expected time.
Then, for instance, be a pensions fund management scheme that raises the assets value by a positive constant $\theta_n = \theta$, when the assets value falls equal to the liabilities process in the n^{th} time. This corresponds to consider a modification in the process A(t), we call it $\bar{A}(t)$, that now restarts at instants T_n, when hits the barrier $L(t)$, by the n^{th} time at level $L(T_n) + \theta_n$.
A convenient choice for management policy is, for example, one in which:

$$\theta_n = L(T_n)\left(e^{\theta} - 1\right), \text{for some } \theta > 0 \tag{13}$$

Then, the corresponding modification of $Y(t)$, $\bar{Y}(t)$, behaves as a generalized Brownian motion process that restarts at the level $\theta = \ln\frac{L(T_n)+\theta_n}{L(T_n)}$ when it its 0, at times T_n.
The **present value of the cost of perpetual maintenance of the pension fund,** see (Ferreira and Filipe, 2021, 2021a), is considered, due to the proposed asset-liability management scheme, given by the random variable:$V(r, a, \theta) = \sum_{n=1}^{\infty} \theta_n e^{-rT_n} = \sum_{n=1}^{\infty} b\left(e^{\theta} - 1\right) e^{-(r-\rho)\,T_n}, r > \rho$, where r represents the due discount rate. This random variable expected value is

$$v_r(a, \theta) = \frac{b(e^{\theta}-1)e^{-K_{r-\rho}a}}{1-e^{-K_{r-\rho}\,\theta}},\; K_i = \frac{\mu+\sqrt{\mu^2+2i\sigma^2}}{\sigma^2} \tag{14}$$

Note that $\lim\limits_{\theta\to 0} v_r(a, \theta) = \frac{be^{-K_{r-\rho}a}}{K_{r-\rho}}$.

6 CONCLUSIONS

These results were obtained considering the simple and general random walk that are classic and widely studied stochastic processes. Since its general ideas are easily grasped by everyone, which quickly connect them with real systems, the random walk is used to model situations more disparate realities, far beyond the reserve evolution models considered in this work. They are also, and very often, used to build other complex systems, sometimes much more complex, for other types of systems.
We highlight in our approach a set of different methodologies that were applied to the study of this type of processes such as the cases of Difference Equations and Martingales Theory.
In our approach, reserves systems are treated as physical systems. In our study, we recognize that this may be a limitation to be considered, since it is not obvious that the direct application of these principles to financial reserve funds can be legitimate when their own dynamics of appreciation and devaluation over time are ignored.
The models themselves, and the consequent valuation of stability systems based on the assessment of the probability of depletion or ruin of reserves, seem to be valid only in constant price contexts.
But, in section 5. we consider the integration of factors that are associated with the process of temporal depreciation of the value of money while considering the modeling of financial reserves, when dealing with possible politics aiming to guarantee the fund sustainability.

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