Specially Structured Flow Shop Scheduling Models with Processing Times as Trapezoidal Fuzzy Numbers to Optimize Waiting Time of Jobs

Bharat Goyal and Sandeep Kaur

January 4, 2021

Specially Structured Flow Shop Scheduling Models with processing times as Trapezoidal Fuzzy Numbers to optimize Waiting time of Jobs

Bharat Goyal[1] and Sandeep Kaur[2]

[1]Department of Mathematics, G.S.S.D.G.S. Khalsa College, Patiala (Punjab), India
[2] Research Scholar, Department of Mathematics, Punjabi University Patiala (Punjab), India
bhartu89@gmail.com

Abstract. This paper presents two-stage flow shop fuzzy scheduling approach under uncertain situations. The processing times are demonstrated by trapezoidal membership function. An exact algorithm is proposed with an objective to achieve a schedule that minimizes the total waiting time of jobs in specially structured model where the AHR of processing times is not on the whole arbitrary but must satisfy a definite condition. Most of the literature in scheduling focuses on to minimize the make span. Significance of the desired objective and effectiveness of proposed algorithm is exhibited in comparison to Johnson [7], Palmer [13], NEH [12] and Nailwal K.K. et. al. [8] and Goyal B. et. al. [1] Heuristic approaches. The results obtained shows the best out of the five as well whenever objective of minimizing waiting times is concerned.

Keywords: Flow Shop Scheduling, Trapezoidal Fuzzy Numbers, Heuristic, Total Waiting Time, Job Sequencing.

1 Introduction

Scheduling is a deliberate study in decision making problems. Flow shop scheduling is selection making ideology which is used in present time engineering and industrial manufacturing services. Job-shop scheduling model comprises various jobs along- with some operations which are to be performed on different machines. The machines can be railway tracks, ophthalmologist, machines in the car manufacturing industry and many other. Jobs can be arrival and departure of trains, diagnosis of patients and assembling of car parts in sequence respectively. Every job has been processed on machines for certain time period. The processing times of different operations of a job do not intersect with each other. One machine can implement only one job at a time. Flow shop scheduling problem is one of the most prominent problem in field of scheduling. m-operations of each job must be performed in same order on m different machines. For the permutable theories and heuristic approaches, scheduling becomes an integral part as it provides various techniques to achieve the objective. Scheduling aims to meet one or more objective by performing various jobs over available machines.

Most of the literature deals with deterministic processing times but in real world there are a lot of problems that have uncertain situations. Approaches that deals with exact processing times fail to tackle with uncertainty-based issues. To overcome such indeterminist problems scheduling approaches, take advantage of fuzzy environment as fuzzy environment provides solutions for uncertainty-based problems. Trapezoidal fuzzy membership functions can be used to demonstrate this vague information. The objective of obtaining an optimal or near optimal solution to minimize make span has been the key interest of almost every researcher in scheduling theory. In this paper we propose an exact method to obtain an optimal sequence to minimize total waiting time of jobs. McCahon and Lee [11] proposed an algorithm with generalized mean values (GMVs) in order to defuzzify the fuzzy numbers with triangular membership. Later on improved results were obtained by Sanja and Xueyan [14] who made use of α-cut approach to minimize the make span in two machine flow shop scheduling problem.

Werner Van Leekwijck& Etienne E. Kerre [9] studied various defuzzification techniques and find that the maxima methods give satisfactory results referring to the primary defuzzification methods. To get the optimal results, Yager's ranking method [16] is used in this paper.

First optimal two and three stage scheduling approach was originated by Johnson [7] around 1950s to optimize the makespan. Palmer [13] applied the heuristic approach for minimizing make-span in n-job m -machine problem. Nawaz et.al. [12] introduced Nawaz, Enscore and Ham (NEH) algorithm based on heuristic approach for reducing total processing time on all machines. Also, Chakraborty, U. K. &Laha, D. [3] attempted to obtain a good solution in polynomial time by modifying NEH algorithm. Szwarc, W. [15] surveyed all significant acquainted cases of the m×n flow shop problem and provides optimal results for three new cases. Further Gupta, J. N. D. [6] consider the specially structured models in flow shop scheduling to reduce the makespan. Apart from this, numerous heuristic approaches were made alike of Bhatnagar V., Das, G., & Mehta, O.P. [2] and Gupta, D. et.al. [5] to optimize the waiting time of jobs with deterministic processing times. Maggu, P. L., & Das, G. [10] studied scheduling models with various objectives and parameters. Incorporating the concept of job block and transportation time Gupta, D. & Goyal, B. [4] obtained optimum total waiting time of jobs in two stage flow shop scheduling problem. Nailwal K.K.et. al. [8] has developed a Heuristic Approach to obtain a sequence of jobs to minimize the total elapsed time when there is lack of intermediate storage between the processing of jobs. Goyal B. [1] et.al. proposed a heuristic approach to minimize the waiting time of jobs when the processing times are random. This paper aims to propose a specially structured problem, with two machines n-jobs flow shop scheduling, to minimize total waiting time of jobs in fuzzy environment.

2 Preliminaries

2.1 Fuzzy Number: A fuzzy number $\widetilde{N}$ is a convex fuzzy set of the real line R along with its membership function $\mu_{\widetilde{N}}: R \to [0,1]$ satisfies the following axioms:
(i) $\widetilde{N}$is normal i.e. there exists exactly one $x \in R$ for which $\mu_{\widetilde{N}}(x) = 1$.
(ii) $\mu_{\widetilde{N}}(x)$ is piecewise continuous.

2.2 Trapezoidal Fuzzy Number: A fuzzy number $A = (a_1, a_2, a_3, a_4)$is said to be a trapezoidal fuzzy number if its membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 < x < a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 < x < a_4 \\ 0, & x > a_4 \end{cases} \quad (1)$$

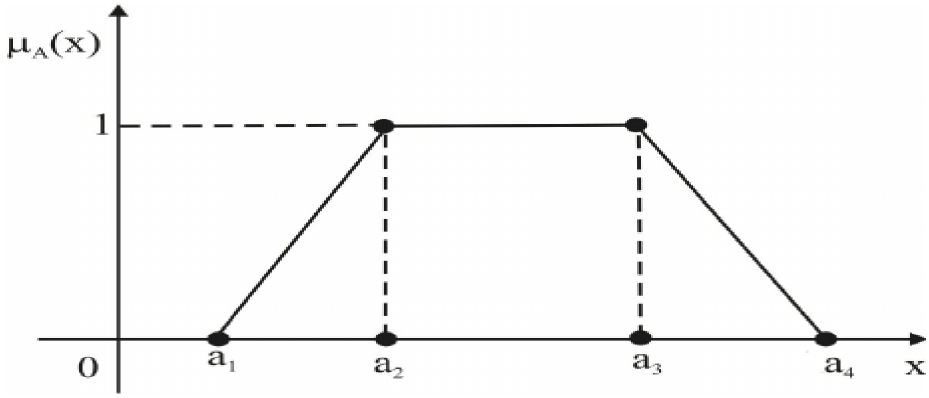

Fig.1. Trapezoidal Membership Fuzzy Number $A = (a_1, a_2, a_3, a_4)$

2.3. Yager's Ranking Method

For a trapezoidal fuzzy number$\tilde{F}$, Yager's Ranking index [16] is given by

$$R(\tilde{F}) = \frac{1}{2}\int_0^1 (F_\alpha^l + F_\alpha^u)\, d\alpha \qquad (2)$$

where (F_α^l, F_α^u) is the α-level cut for the fuzzy number $\tilde{F}$, $R(\tilde{F})$ is the Yager's ranking index for fuzzy number $\tilde{F}$.

2.4 Waiting time of jobs

The waiting time U_β of a job β in a flow-shop scheduling problem is defined as the time which is consumed on waiting in queue for processing on second machine.

2.5 Total waiting time of jobs

The total waiting time W_t can be stated as the sum of all waiting times i.e.

$$W_t = \sum_{i=1}^{n} U_{\beta_i} \qquad (3)$$

3 Format of Framework

3.1 Notation: Different notations used in the paper are as follows:

Notations	**Explanation**
$\boldsymbol{i}$	index for jobs β_i i=1,2,3,…,n
$\boldsymbol{f_i^M}$	Fuzzy processing time of job *i* on machine M
$\boldsymbol{p_i^M}$	AHR value of fuzzy processing time of job i on machine M
$\boldsymbol{C_\beta^M}$	Completion time of job β on machine M
$\boldsymbol{U_\beta}$	Time consumed on waiting by job β
$\boldsymbol{Y_i^M}$	starting time of job i on machine M
$\boldsymbol{W_t}$	Total waiting time of jobs

3.2 Postulates

1. At the initial time t=0, all machines are ready to perform their tasks (jobs).
2. Whichever job is to be processed on first machine, it is always available.
3. Every machine is available without any halt and failure meanwhile the scheduling process.
4. Machines set up time is assumed to be included in processing times.

3.3 Problem Description

Let n-jobs are carried upon two machines (Machine1 and Machine2) in the flow shop process with processing time of i-th job on machine M, $(M = 1,2)$ taken as trapezoidal fuzzy numbers are denoted as f_i^M. Mathematically the problem description can be framed as represented in the following table:

Table 1. Problem description in matrix form.

Job i	Machine 1 $\mathbf{f_i^1}$	Machine 2 $\mathbf{f_i^2}$
1	$(\alpha_{11}^1, \alpha_{21}^1, \alpha_{31}^1, \alpha_{41}^1)$	$(\alpha_{11}^2, \alpha_{21}^2, \alpha_{31}^2, \alpha_{41}^2)$
2	$(\alpha_{12}^1, \alpha_{22}^1, \alpha_{32}^1, \alpha_{42}^1)$	$(\alpha_{12}^2, \alpha_{22}^2, \alpha_{32}^2, \alpha_{42}^2)$
3	$(\alpha_{13}^1, \alpha_{23}^1, \alpha_{33}^1, \alpha_{43}^1)$	$(\alpha_{13}^2, \alpha_{23}^2, \alpha_{33}^2, \alpha_{43}^2)$
.	.	.
.	.	.
.	.	.
n	$(\alpha_{1n}^1, \alpha_{2n}^1, \alpha_{3n}^1, \alpha_{4n}^1)$	$(\alpha_{1n}^2, \alpha_{2n}^2, \alpha_{3n}^2, \alpha_{4n}^2)$

The Yager's Ranking index of processing times p_i^M(M=1, 2)are satisfying the condition

$$\max p_i^1 \leq min\, p_j^2 \tag{4}$$

The objective is to obtain the best schedule in order to minimize the total waiting time.

3.4 Significance

The proposed work deals with the objective of minimizing the total waiting time of jobs under Fuzzy environment with processing times as Trapezoidal Fuzzy numbers. Most of the research work in scheduling focuses on to minimize the total elapsed time and many Heuristic approaches has been developed like NEH [12], Palmer [13] , Nailwal [8] to minimize it. The proposed problem has the significant objective and new approach in the sense that as most of the research only suggests to minimize the cost of Industries but customer satisfaction is also one of the important issue in Today's competitive market. The objective of minimizing the waiting time of jobs will be of great significance if Industrial manager has contract with the consumer to complete their job without too much waiting for the subsequent processing.

4 Theorems and Results

Theorem 4.1: Let n −jobs 1, 2, …, n be processed on two machines (Machine1 and Machine2) in flow shop process without fleeting and satisfying the structural relationship

$$\max\, p_i^1 \leq min\, p_j^2 \tag{4}$$

where p_i^M is the Yager's ranking index[16]value of the equivalent fuzzy processing time required by job i on machine M, $(M = 1,2)$: $(i, j = 1,2,3, \dots, n)$, then W_t, the total waiting time of jobs is given by

$$W_t = n\, p_{\beta_1}^1 + \sum_{q=1}^{n-1}(n-q)V_{\beta_q} - \sum_{j=1}^{n} p_{\beta_j}^1 \tag{5}$$

where,

$$V_{\beta_q} = (p_{\beta_q}^2 - p_{\beta_q}^1) \tag{6}$$

Proof: Firstly C_β^M , the completion time of job β on machine M will be evaluated,
For the sequence, $S = \beta_1, \beta_2, \beta_3, \dots, \beta_k, \dots, \beta_n$

Claim:

$$C^2_{\beta_n} = p^1_{\beta_1} + p^2_{\beta_1} + p^2_{\beta_2} + \cdots + p^2_{\beta_n} \quad (7)$$

Applying mathematical induction on n.

Let the statement $P(n)$:

$$C^2_{\beta_n} = p^1_{\beta_1} + p^2_{\beta_1} + p^2_{\beta_2} + \cdots + p^2_{\beta_n} \quad (7)$$

Now, for $n = 1$,

$$C^2_{\beta_1} = p^1_{\beta_1} + p^2_{\beta_1} \quad (8)$$

Now, let for $n = k, P(k)$be true

Then for $P(k+1)$, using (4)

$$C^2_{\beta_{k+1}} = \max\left(C^1_{\beta_{k+1}}, C^2_{\beta_k}\right) + p^2_{\beta_{k+1}} \quad (9)$$

Proving,

$$C^2_{\beta_{k+1}} = p^1_{\beta_1} + p^2_{\beta_1} + p^2_{\beta_2} + \cdots + p^2_{\beta_k} + p^2_{\beta_{k+1}} \quad (10)$$

Secondly U_β, the time consumed on waiting by job β will be evaluated

Claim: For the sequence $S = \beta_1, \beta_2, \beta_3, \dots, \beta_k, \dots, \beta_n$ of jobs

$$U_{\beta_k} = p^1_{\beta_1} + \sum_{q=1}^{k-1} V_{\beta_q} - p^1_{\beta_k}, \quad k = 2,3, \dots, n \quad (11)$$

Obviously

$$U_{\beta_1} = 0 \quad (12)$$

and

$$U_{\beta_k} = Y^2_{\beta_k} - C^1_{\beta_k}, \qquad k = 2,3, \dots., n \quad (13)$$

implies,

$$U_{\beta_k} = \max\left(C^2_{\beta_{k-1}}, C^1_{\beta_k}\right) - C^1_{\beta_k}, \quad k = 2,3, \dots., \mathrm{n} \quad (14)$$

According to the condition (4) of specially structured model we have

$$U_{\beta_k} = p^1_{\beta_1} + \sum_{q=1}^{k-1} V_{\beta_q} - p^1_{\beta_k}, \quad k = 2,3, \dots, n \quad (15)$$

Approaching to the main proof of the theorem

$$W_t = U_{\beta_1} + U_{\beta_2} + U_{\beta_3} + \cdots + U_{\beta_n} \quad (16)$$

$$W_t = n\, p^1_{\beta_1} + \sum_{q=1}^{n-1} (n-q) V_{\beta_q} - \sum_{j=1}^{n} p^1_{\beta_j} \quad (17)$$

Theorem 4.2: For a natural number k and real numbers $v_1, v_2, \dots, v_k$. Then of all the linear combination of the form

$$\sum_{i=0}^{k-1} (k-i) v_{i+1},$$

the one is minimum if

$$v_1 \le v_2 \le \ldots \le v_k$$

Proof: Applying induction hypothesis on k

The result holds trivially for $k = 1$.

Assume that the result comes true for less than k real numbers.

Now for

$$v_1 \le v_2 \le \ldots \le v_k$$

$$k v_1 + (k-1) v_2 + (k-2) v_3 + \ldots + 2 v_{k-1} + v_k$$

$$= (k-1) v_1 + (k-2) v_2 + (k-3) v_3 + \ldots + v_{k-1} + \sum_{i=1}^{k} v_i$$

As last term $\sum_{i=1}^{k} v_i$ is constant, therefore hypothesis assumption implies

$$k v_1 + (k-1) v_2 + (k-2) v_3 + \ldots + 2 v_{k-1} + v_k$$

is minimum.

Remark: Based on the result from theorem 4.2, we observe that for a n-job sequence $S: \beta_1, \beta_2, \ldots, \beta_n$, the term

$$\sum_{q=1}^{n-1}(n-q)V_{\beta_q}$$

in equation (5) will be minimum if n-jobs in sequence S are arranged in non-decreasing order of the values

$$V_{\beta_q} \text{ and } \sum_{j=1}^{n} p_{\beta_j}^1$$

is constant for every sequence of jobs. In keeping mind these observations, an exact method is proposed in section 5 to minimize the total waiting time W_t for two-machine specially structured flow-shop scheduling problems.

5 Algorithm

The proposed algorithm involves the following steps:
Step 1. Compute the Ranking Index value of fuzzy processing time $f_i^M = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for all jobs $j_i, i = 1,2,3, \ldots, n$ by using the Yager's Ranking Index [16]
Step 2. Check the structural condition i.e. max $p_i^1 \leq min\, p_j^2$
Step 3. Compute

$$d_{i_q} = (n-q)V_i$$

where

$$V_i = p_i^2 - p_i^1$$

for i= $1,2,3, \ldots, n-1$ and get the computed entries in the following tabulated format

Table 2. Format of the computed entries

Job i	**Machine 1** p_i^1	**Machine 2** p_i^2	V_i $p_i^2 - p_i^1$	$d_{i_q} = (n-q)V_i$ $q=1$	$q=2$	$q=3$	…	$q=n-1$
1	p_1^1	p_1^2	V_1	d_{1_1}	d_{1_2}	d_{1_3}	…	$d_{1_{n-1}}$
2	p_2^1	p_2^2	V_2	d_{2_1}	d_{2_2}	d_{2_3}	…	$d_{2_{n-1}}$
3	p_3^1	p_3^2	V_3	d_{3_1}	d_{3_2}	d_{3_3}	…	$d_{3_{n-1}}$
.	.	.	.	.	.	.		.
.	.	.	.	.	.	.	…	.
.	.	.	.	.	.	.		.
n	p_n^1	p_n^2	V_n	d_{n_1}	d_{n_2}	d_{n_3}	…	$d_{n_{n-1}}$

Step 4. Arranging the jobs in ascending order of V_i and get the sequence $S_1 = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\}$
Step 5. Locate minimum of processing time of machine 1 and call it p_x^1. Further, check the condition

$$p_x^1 = p_{\beta_1}^1,$$

If this condition met then the sequence obtained in previous step is optimal otherwise go to next step.
Step 6. Now obtain other sequences $S_i, i = 2,3,4, \ldots, n$ by interchanging the i_{th}job with first one of the sequence S_{i-1} and keeping the rest of the job sequence unaltered.

Step 7. Calculate the total waiting time W_t for all the sequences $S_1, S_2, S_3, \ldots, S_n$ using formula defined in equation (5)
Step 8. Pick the sequence with minimum total waiting time from the list mentioned in previous step and this is the required optimal sequence.

6 Numerical Illustration

To evaluate the performance of the solution method of proposed algorithm,a numerical illustration of randomly generated problem with five jobs and two machines is described below:
Let ten jobs 1, 2, 3, 4, 5, 6, 7, 8, 9, 10(say) are carried upon two machines (Machine 1 and Machine 2)

Table 3. Fuzzy processing times for jobs

Job i	Machine 1 f_i^1	Machine 2 f_i^2
1	(65,69,77,93)	(75,89,97,112)
2	(61,72,83,93)	(80,92,104,106)
3	(65,70,84,87)	(81,86,95,106)
4	(64,71,79,94)	(76,89,99,107)
5	(57,75,78,88)	(79,83,98,107)
6	(54,71,76,92)	(82,87,102,113)
7	(65,72,85,89)	(76,85,103,110)
8	(60,70,80,92)	(80,87,98,112)
9	(58,69,78,90)	(76,84,94,106)
10	(63,69,84,87)	(75,89,94,107)

Yager's Ranking index [16] of above mentioned fuzzy processing times are represented in the table below

Table 4. Crisp values of fuzzy processing times

Job i	Machine 1 p_i^1	Machine 2 p_i^2
1	76.00	93.25
2	77.25	95.50
3	76.50	92.00
4	77.00	92.75
5	74.50	91.75
6	73.25	96.00
7	77.75	93.50
8	75.50	94.25
9	73.75	90.00
10	75.75	91.25

It can be seen that max $p_i^1 \le min\, p_j^2$ so the structural condition is met.

According to step 4 we get the sequence

$$S_1 = \{\beta_3, \beta_{10,}\beta_4, \beta_7, \beta_9, \beta_1, \beta_5, \beta_2, \beta_8, \beta_6\}$$

Since $p_x^1 \neq p_\mu^1$, so all the possible produced sequences according to step 6 are

$$S_2 = \{\beta_{10}, \beta_{3,}\beta_4, \beta_7, \beta_9, \beta_1, \beta_{5,}\beta_2, \beta_8, \beta_6\};$$

$$S_3 = \{\beta_4, \beta_{3,}\beta_{10}, \beta_7, \beta_9, \beta_1, \beta_{5,}\beta_2, \beta_8, \beta_6\};$$

$$S_4 = \{\beta_7, \beta_{3,}\beta_{10}, \beta_4, \beta_9, \beta_1, \beta_{5,}\beta_2, \beta_8, \beta_6\};$$
$$S_5 = \{\beta_9, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_1, \beta_{5,}\beta_2, \beta_8, \beta_6\};$$
$$S_6 = \{\beta_1, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_9, \beta_{5,}\beta_2, \beta_8, \beta_6\};$$
$$S_7 = \{\beta_5, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_9, \beta_{1,}\beta_2, \beta_8, \beta_6\};$$
$$S_8 = \{\beta_2, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_9, \beta_{1,}\beta_5, \beta_8, \beta_6\};$$
$$S_9 = \{\beta_8, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_9, \beta_{1,}\beta_5, \beta_2, \beta_6\};$$
$$S_{10} = \{\beta_6, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_9, \beta_{1,}\beta_5, \beta_2, \beta_8\}$$

Table 5. Optimal schedule table

Sequence	Total Waiting Time(W_t)
S_1	**733.25**
S_2	**725.75**
S_3	**738.75**
S_4	**746.25**
S_5	**708.25**
S_6	**735.75**
S_7	**720.75**
S_8	**755.25**
S_9	**741.75**
S_{10}	**755.25**

Thus $\min\{W_t\} = 708.25$ units of time corresponding to the sequence $\boldsymbol{S_5}$
Hence $S_5 = \{\beta_9, \beta_{3,}\beta_{10}, \beta_4, \beta_7, \beta_1, \beta_{5,}\beta_2, \beta_8, \beta_6\}$is the desired optimal schedule of jobs having optimal waiting time of jobs.

7 Computational Analysis

To look over the suitability of the proposed heuristic, numerous examples of various groups are randomly generated in which each group varies upon different number of jobs. Here ten groups are generated with job sizes 5, 10, 15, 20, 30, 40, 50, 55, 60, 80 and each group is studied over 10 different randomly generated problems with processing times as trapezoidal fuzzy numbers. For all groups, mean of the total waiting time of each problem for proposed algorithm is compared with the mean of already existed make-span approaches of Johnson [7], Palmer [13] ,NEH [12], Nailwal [8] and waiting time approach of B. Goyal [1] and are plotted in graph as shown in Fig. 2, which demonstrate that the curve of Palmer [13] is high among all. Furthermore, the curve of NEH [12] and B.Goyal [1] is closer than others to the proposed algorithm's curve

Table 6. Mean of the total waiting times

n	Palmer [13]	Johnson [7]	NEH [12]	Proposed Algorithm	Nailwal [8]	B.Goyal [1]
5	186.13	181.15	170.95	168.25	185.15	168.70
10	843.48	824.65	769.67	754.77	822.85	755.60
15	2003.33	1951.75	1810.05	1780.28	1918.33	1782.03
20	3748.40	3663.90	3323.13	3270.68	3565.30	3273.07
30	8605.92	8420.77	7594.57	7508.63	8170.70	7528.27
40	15177.85	14894.48	13399.88	13295.23	14299.75	13308.53
50	23707.67	23245.47	20947.58	20777.95	22445.40	20807.03
55	29029.67	28534.03	25515.53	25321.40	27339.42	25339.42
60	34318.65	33700.85	30345.05	30155.90	32393.00	30176.85
80	61928.07	60686.47	54451.72	54161.63	58365.22	54202.15

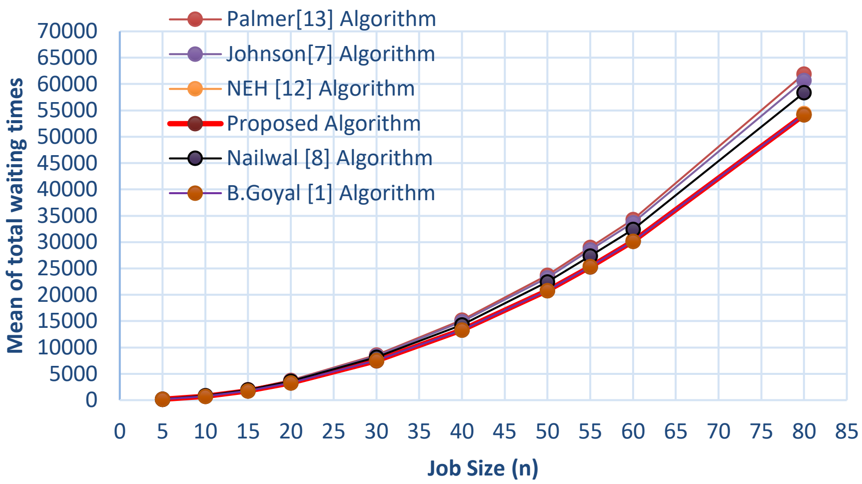

Fig. 2. Comparison of Computational results

In addition, the percentage of error for each of the problem is also calculated by using the formula

$$e_{rr} = [(W_\delta - W_\Theta)/W_\Theta] * 100$$

whereW_δ is the total waiting time of existed algorithms and W_Θ is the total waiting time of the same job computed by using proposed algorithm. For the sake of measuring the wellness of the proposed algorithm, mean of percentage error is calculated for all job groups and then figured out in the graph below, shown in Fig. 3.

Table 7. Mean of percentage errors

n	Palmer [13] Algorithm	Johnson [7] Algorithm	NEH [12] Algorithm	Nailwal [8] Algorithm	B. Goyal [1] Algorithm
5	**10.73**	**7.73**	**1.61**	**10.08**	**0.27**
10	**11.78**	**9.28**	**1.97**	**9.08**	**0.11**
15	**12.55**	**9.65**	**1.67**	**7.76**	**0.10**
20	**14.62**	**12.05**	**1.61**	**9.01**	**0.07**
30	**14.63**	**12.17**	**1.15**	**8.83**	**0.26**
40	**14.20**	**12.07**	**0.79**	**7.58**	**0.10**
50	**14.13**	**11.91**	**0.82**	**8.04**	**0.14**
55	**14.66**	**12.71**	**0.77**	**7.97**	**0.07**
60	**13.82**	**11.77**	**0.63**	**7.43**	**0.07**
80	**14.34**	**12.04**	**0.54**	**7.77**	**0.07**

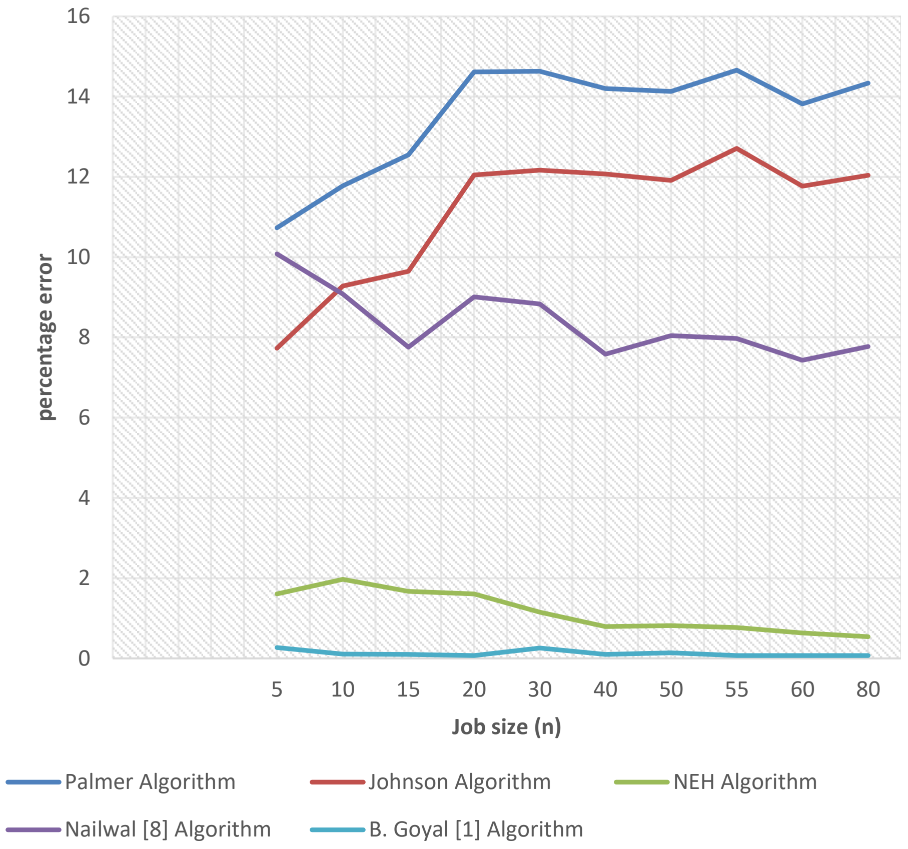

Fig. 3. Average percentage error in the computational experiments

Fig. 3 shows that the Johnson [7], Palmer [13] and NEH [12], Nailwal [8] and B.Goyal [1] algorithms considered for comparison in this paper yields large total waiting time than the proposed one. Also, the error curve shows that NEH [12] algorithm with make-span approach and B. Goyal [1] returns less total waiting time than the Johnson [7], Palmer [13] and Nailwal[8] algorithm.

From the computational experiments, it is noted that the error is independent of job sizes as it can be seen in table 7, that group with 10 jobs has mean of percentage errors as 11.78 units in Palmer's algorithm and it increases with the increase in number of job. When job size is increased to 30 with another data set of problems, it rises to 14.63 units. But for the job size 40, it reduces to 14.20 units, it again increases to 14.66 when size of jobs is 55. This shows that error is independent of job size but it depends upon the choice of randomly generated fuzzy processing times. A key point is also noted that the mean error increases and decreases in the same manner for both the Palmer and Johnson's algorithm for different job groups but this is not so in the case of NEH algorithm.

Table 8. Average of mean percentage errors

Algorithm	Average of mean percentage errors
Palmer [13]	**13.55**
Johnson [7]	**11.14**
NEH [12]	**1.16**
Nailwal[8]	**8.35**
B. Goyal [1]	**0.12**

Furthermore, it can be seen from table 8 that NEH [12] and B. Goyal[1] algorithm is very close to the exact solution whereas Palmer [13] algorithm produces an error significantly larger than the Johnson [7] algorithm. Also, the significant less error in-makespan approach of NEH [12] algorithm clarifies that the algorithm produces a near optimal solution to minimize the idle time of jobs as well.

8 Conclusion

In this paper, an exact algorithm is established to achieve the aim of minimizing total waiting time of jobs but there may be some possibilities that make-span or other costs such as machine idle cost etc. may increases. From the commercial point of view, it is the primary need in the industries, when industry's manager has promise with the consumer to make their wait as less as possible for completing a project. The computational experiments manifest the propriety of proposed algorithm when compared with the existing approaches for make-span made by Johnson [7], Palmer[13] and Nawaz, Enscore and Ham (NEH)[12], Nailwal [8], and waiting time approach by B. Goyal[1]. Further it can be concluded that NEH [12] algorithm minimizes the make-span by reducing the idle time of jobs consumed in queue for processing on second machine and B. Goyal [1] produces a schedule of jobs that produces near to the optimal solution in specially structured problem of minimizing the total waiting time of jobs. The present work can be enhanced by taking setup times for machines, making use of trapezoidal fuzzy numbers or by taking three or more machines.

References

1. B. Goyal and S. Kaur.: Minimizing waiting time of jobs in flow-shop scheduling: A heuristic approach. Materials Today: Proceedings (2020). https://doi.org/10.1016/j.matpr.2020.09.797
2. Bhatnagar V., Das, G., & Mehta, O.P.: n-job two machine flow-job shop scheduling problem having minimum total waiting time for all jobs. PAMS, X, 1-2 (1979).
3. Chakraborty, U. K., &Laha, D.: An improved heuristic for permutation flow-shop scheduling. International Journal of Information and Communication Technology 1, 89-97 (2007).
4. Gupta, D. & Goyal, B.: Specially Structured Flow Shop Scheduling in Two Stage with Concept of Job Block and Transportation Time to Optimize Total Waiting Time of Jobs. International journal of Engineering & Technology 10(5), 1273-1284 (2018).
5. Gupta, D., &Goyal, B.: Waiting times of jobs models in $n \times 2$ flow shop scheduling. Lambert Academic Publishing, Mauritius (2020)
6. Gupta, J. N. D.: Optimal schedules for special structure flowshops. US. Postal Service, Washington, D. C., 255-269(1975).

7. Johnson, S. M.: Optimal two- and three-stage production schedules with setup times included. Naval Research Logistics Quarterly 1(1), 61-68(1954).
8. Nailwal K.K., Gupta D., KawalJeet.: Heuristics for no-wait flow shop scheduling problem. Internatonal Journal of Industrial Engineering Computations 7, 671-680 (2016)
9. Leekwijck, W. V., &Kerre, E. E.: Defuzzication: criteria and classication. Fuzzy Sets and Systems 108, 159–178 (1999).
10. Maggu, P. L. & Das, G.: Elements of Advanced Production Scheduling. United Publishers and Periodical Distributors, New Delhi (1985).
11. McCahon, S. & Lee, E.S.: Job sequencing with fuzzy processing times. Computer and Mathematics with applications 19(7), 31-41(1990).
12. Nawaz, M., Enscore, Jr. EE. &Ham, I.: A heuristic algorithm for the m-machine, n-job flow shop sequencing problem. OMEGA. The international Journal of Management Science 11(1), 91-95 (1983).
13. Palmer, D.S.: Sequencing jobs through a multi stage process in the minimum total time- A quick method for obtaining a near optimum. Operations Research 16, 101-107(1956).
14. Sanja, P. &Xueyan, S.: A new approach to two-machine flow shop problem with uncertain processing times. Optimization & Engineering 7(3), 329-343(2006).
15. Szwarc, W.: Special cases of the flow shop problem. Naval Research Logistics Quarterly 24(3), 483-492 (1977).
16. Yager, R.R.: A procedure for ordering fuzzy subsets of the unit interval. Information Sciences 24, 143-161 (1981).