# Quadratics Classification

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# Quadratics Classification

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**Abstract:** This paper introduces a new classification criterion for Quadratics.
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## 1 Introduction

In the Offset in Quadratics study, we have determined our simplest equation generated from 3 elements $(x_1, x_2, x_3)$ given by any quadratic $Y[\mathrm{y}] = ay^2 + by + c$ as being

$$Y[\mathrm{y}] = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{x_3 - x_1}{2}\right)y + x_2$$

Now let's check what happens when two of these 3 elements are equal. Then, we will study the results and we will classify the quadratics into 3 fundamental types.

### 1.1 Previous conventions:

Because our tables will show vertical sequences where the indexes will be on vertical and because on vertical, we have Y-axis in the XY-plane, so the sequences integers elements have to appear in X-axis as a function of the Y-axis. Due to that, in all these studies we will represent any polynomial equation as being in the function of $y$, or just function $Y[y]$, or $x = Y[y]$.

### 1.2 Notation for Polynomials In these studies

Generically we will denote any polynomial element as being $Y[y]$. When we want to draw the polynomial in the XY-plane we will make $x$ in the function of $y$. In the cartesian plane (square lattice grid) we can consider $x = Y[y]$. In different grid other than cartesian plane $x \neq Y[y]$.

When we want to distinguish the $d^{\text{th}}$-degree of the polynomial, we will notate $Yd[y]$ or $x = Yd[y]$.

When we want to make a $p^{\text{th}}$-power operation on an $d^{\text{th}}$-degree polynomial, we will notate: $(Yd[y])^p$.

- Constant (polynomial degree 0) will be noted as

$$Y0[y] = c$$

- Linear (polynomial 1st-degree) will be noted as

$$Y1[y] = by + c$$

- Quadratic (polynomial 2nd-degree) will be noted as

$$Y2[y] = ay^2 + by + c$$

- Cubic (polynomial 3rd-degree) will be noted as

$$Y3[y] = a_3y^3 + ay^2 + by + c$$

- Quartic (polynomial 4th-degree) will be noted as

$$Y4[y] = a_4y^4 + a_3y^3 + ay^2 + by + c$$

- Quintic (polynomial 5th-degree) will be noted as

$$Y5[y] = a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c$$

And so on for Sextic, Septic, Octic, Nonic, Decic, etc.

Generic equation of polynomial $d^{\text{th}}$-degree:

$$Yd[y] = a_dy^d + a_{d-1}y^{d-1} + \cdots + a_4y^4 + a_3y^3 + ay^2 + by + c$$

Generically, to be used in any recurrence equation, we will adopt these equalities notation:

$$\begin{aligned} Yd[-3] &= e \\ Yd[-2] &= f \\ Yd[-1] &= g = x_1 \\ Yd[0] &= h = x_2 \\ Yd[1] &= i = x_3 \\ Yd[2] &= j \\ Yd[3] &= k \end{aligned}$$

## 1.3 Notation for index direction in any polynomial sequence (to be used in recurrence equations)

Any polynomial Integer sequence has 2 directions. This is the reason any polynomial has 2 recurrence equations. So, if the direction is given by

$$Yd[y] \equiv (\dots, e, f, g, h, i, j, k, \dots) = \backslash(\dots, k, j, i, h, g, f, e, \dots)\backslash$$

then, the reversal direction will be given by

$$\backslash Yd[y]\backslash \equiv (\dots, k, j, i, h, g, f, e, \dots) = \backslash(\dots, e, f, g, h, i, j, k, \dots)\backslash$$

## 1.4 Inflection point vs. vertex nomenclature

Because of the definition of the inflection point is in differential calculus *"an inflection point, point of inflection, flex, or inflection (British English: inflexion[citation needed]) is a point on a continuous plane curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa*"; and

Because of the definition of the vertex in geometry as being "*In geometry, a vertex (plural: vertices or vertexes) is a point where two or more curves, lines, or edges meet. As a consequence of this definition, the point where two lines meet to form an angle and the corners of polygons and polyhedra are vertices*";

Because *"In the geometry of planar curves, a vertex is a point of where the first derivative of curvature is zero*";

And like all studies between polynomials, no feature or phenomenon indicates that there is a difference in behavior between quadratic and other polynomial orders, then, there is no reason to differentiate the inflection point phenomena in quadratics from other polynomials. So, there is no reason to have different names.

In these studies, we will refer to this phenomenon in our tables, text, and figures as being only inflection point, even in quadratics which usually has the usual vertex name. Moreover, higher degrees of polynomials then quadratics, besides inflection point may have two or more turning points. But, the common phenomenon among all polynomials is the inflection point.

The definition of a single Inflection Point nomenclature in common to all polynomials becomes important when we compare the behavior of the offset at all degrees.

In these studies, the coordinates of an inflection point in XY-plane will be given by $x_{ip}$ and $y_{ip}$. Also, we will denote an inflection point as being $ip(x_{ip}, y_{ip})$.

## 2 Six equalities to cover all quadratics classification

From our general equation $Y[y] = \mathrm{a}y^2 + by + c = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{x_3 - x_1}{2}\right)y + x_2$ we have 3 elements $(x_1, x_2, x_3)$ which, to generate a quadratic they cannot all be equal to each other nor $(x_2 - x_1) = (x_3 - x_2)$. This is because the numerator of coefficient $a$ has to be $x_1 - 2x_2 + x_3 \neq 0$.

But nothing prevents them from being with the same value two by two. Let's see the quadratic's behavior in the parameters $a, b, y_{ip}, x_{ip}, \Delta$ for all possible cases:

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 \\ x_1 &= x_3 \\ x_1 &= -x_2 \\ x_2 &= -x_3 \\ x_1 &= -x_3 \end{aligned}$$

### 2.1 The case where $x_1 = x_2$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_3}{2x_1 - 4x_1 + 2x_3} = \frac{x_1 - x_3}{-2x_1 + 2x_3} = -\frac{x_1 - x_3}{2x_1 - 2x_3} = -\frac{1}{2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_1 + x_3}{2} = \frac{-x_1 + x_3}{2} = b$$

$$b = \frac{x_3 - x_1}{2} = a$$

$$\begin{aligned} \Delta &= \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} \\ &= \frac{x_1^2 + 16x_1^2 + x_3^2 - 8x_1x_1 - 8x_1x_3 - 2x_1x_3}{4} = \frac{9x_1^2 + x_3^2 - 10x_1x_3}{4} \\ &= \frac{(-x_1 + x_3)^2 + 8x_1^2 - 8x_1x_3}{4} = \frac{(-x_1 + x_3)^2 - (-8x_1^2 + 8x_1x_3)}{4} \\ &= \frac{(-x_1 + x_3)^2 - 8x_1(-x_1 + x_3)}{4} = (-x_1 + x_3)\frac{(-x_1 + x_3) - 8x_1}{4} \\ &= (-x_1 + x_3)\frac{x_3 - 9x_1}{4} \end{aligned}$$

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{(-x_1 + x_3)\frac{x_3 - 9x_1}{4}}{4\frac{-x_1 + x_3}{2}} = -\frac{\frac{x_3 - 9x_1}{4}}{2} = -\frac{x_3 - 9x_1}{8} = \frac{9x_1 - x_3}{8}$$

$$
\begin{aligned}
x_{ip} &= \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(x_1 - 2x_1 + x_3)} \\
&= \frac{8(-x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(-x_1 + x_3)} \\
&= \frac{-8x_1^2 + 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(-x_1 + x_3)} = \frac{-9x_1^2 + 10x_1x_3 - x_3^2}{8(-x_1 + x_3)} \\
&= \frac{9x_1^2 - 10x_1x_3 + x_3^2}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1^2 - 8x_1x_3}{8(x_1 - x_3)} \\
&= \frac{(x_1 - x_3)^2 + 8x_1(x_1 - x_3)}{8(x_1 - x_3)} = \frac{(x_1 - x_3) + 8x_1}{8} = \frac{9x_1 - x_3}{8}
\end{aligned}
$$

$$
\begin{aligned}
x_{ip} &= x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_1 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_1 + x_3)} = x_1 - \frac{(x_3 - x_1)^2}{8(-x_1 + x_3)} \\
&= \frac{8(-x_1 + x_3)x_1 - (x_3 - x_1)^2}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - (x_3^2 + x_1^2 - 2x_1x_3)}{8(-x_1 + x_3)} = \\
&= \frac{-8x_1^2 + 8x_1x_3 - x_3^2 - x_1^2 + 2x_1x_3}{8(-x_1 + x_3)} = \frac{-9x_1^2 + 10x_1x_3 - x_3^2}{8(-x_1 + x_3)} \\
&= \frac{9x_1^2 - 10x_1x_3 + x_3^2}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1^2 - 8x_1x_3}{8(x_1 - x_3)} \\
&= \frac{(x_1 - x_3)^2 + 8x_1(x_1 - x_3)}{8(x_1 - x_3)} = \frac{(x_1 - x_3) + 8x_1}{8} = \frac{9x_1 - x_3}{8}
\end{aligned}
$$

## 2.2 The case where $x_2 = x_3$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_2}{2x_1 - 2x_2} = \frac{1}{2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - x_2}{2} = -b$$

$$b = \frac{x_3 - x_1}{2} = \frac{x_2 - x_1}{2} = -a$$

$$\begin{aligned}
\Delta = {} & \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_2^2 - 8x_1x_2 - 8x_2x_2 - 2x_1x_2}{4} \\
&= \frac{x_1^2 + 9x_2^2 - 10x_1x_2}{4} = \frac{(x_1 - x_2)^2 + 8x_2^2 - 8x_1x_2}{4} \\
&= \frac{(x_1 - x_2)^2 + 8x_2(x_2 - x_1)}{4} = \frac{(x_1 - x_2)^2 - 8x_2(x_1 - x_2)}{4} \\
&= (x_1 - x_2)\frac{(x_1 - x_2) - 8x_2}{4} = (x_1 - x_2)\frac{x_1 - 9x_2}{4}
\end{aligned}$$

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{(x_1 - x_2)\frac{x_1 - 9x_2}{4}}{4\frac{x_1 - x_2}{2}} = -\frac{x_1 - 9x_2}{8} = \frac{-x_1 + 9x_2}{8}$$

$$\begin{aligned}
x_{ip} &= \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 + x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - 2x_2 + x_2)} \\
&= \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - x_2)} = \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - x_2)} \\
&= \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)(-x_1 + x_2)}{8(x_1 - x_2)} \\
&= \frac{8(x_1 - x_2)x_2 + (x_1 - x_2)(-x_1 + x_2)}{8(x_1 - x_2)} = \frac{8x_2 + (-x_1 + x_2)}{8} = \frac{-x_1 + 9x_2}{8}
\end{aligned}$$

$$\begin{aligned}
x_{ip} &= x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(x_2 - x_1)^2}{8(x_1 - 2x_2 + x_2)} = x_2 - \frac{(x_2 - x_1)^2}{8(x_1 - x_2)} = x_2 + \frac{(x_2 - x_1)^2}{8(x_2 - x_1)} \\
&= x_2 + \frac{x_2 - x_1}{8} = \frac{-x_1 + 9x_2}{8}
\end{aligned}$$

## 2.3 The case where $x_1 = x_3$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = 0$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_2 + x_1}{2} = \frac{2x_1 - 2x_2}{2} = x_1 - x_2$$

$$b = \frac{x_3 - x_1}{2} = \frac{x_3 - x_3}{2} = 0$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_1^2 - 8x_1x_2 - 8x_2x_1 - 2x_1x_1}{4}$$

$$= \frac{16x_2^2 - 16x_1x_2}{4} = 4x_2^2 - 4x_1x_2 = 4x_2(x_2 - x_1)$$

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{4x_2(x_2 - x_1)}{4(x_1 - x_2)} = x_2$$

$$x_{ip} = \frac{8(x_1 - 2x_2 + x_3)x_2 + (x_1 - x_3)^2}{2(x_1 - 2x_2 + x_3)} = \frac{8(2x_1 - 2x_2)x_2}{2(2x_1 - 2x_2)} = \frac{8(2x_1 - 2x_2)x_2}{8(2x_1 - 2x_2)} = x_2$$

$$x_{ip} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(x_1 - x_1)^2}{8(x_1 - 2x_2 + x_1)} = x_2$$

### 2.4 The case where $x_1 = -x_2$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_3}{2x_1 + 4x_1 + 2x_3} = \frac{x_1 - x_3}{6x_1 + 2x_3} = \frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 + 2x_1 + x_3}{2} = \frac{3x_1 + x_3}{2} = b + 2x_1$$

$$b = \frac{x_3 - x_1}{2} = \frac{3x_1 + x_3 - 4x_1}{2} = a - 2x_1$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_1^2 + x_3^2 + 8x_1x_1 + 8x_1x_3 - 2x_1x_3}{4}$$

$$= \frac{25x_1^2 + x_3^2 + 6x_1x_3}{4} = \frac{(3x_1 + x_3)^2 + 16x_1^2}{4}$$

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}}{4\frac{3x_1 + x_3}{2}} = -\frac{\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}}{2(3x_1 + x_3)} == -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)}$$

$$= -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

$$x_{ip} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{-8(x_1 + 2x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(x_1 + 2x_1 + x_3)}$$

$$= \frac{-8(3x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(3x_1 + x_3)}$$

$$= \frac{-24x_1^2 - 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(3x_1 + x_3)} = \frac{-25x_1^2 - 6x_1x_3 - x_3^2}{8(3x_1 + x_3)}$$

$$= -\frac{25x_1^2 + 6x_1x_3 + x_3^2}{8(3x_1 + x_3)} = -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)}$$

$$= -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

$$x_{ip} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = -x_1 - \frac{(x_3 - x_1)^2}{8(x_1 + 2x_1 + x_3)} = -x_1 - \frac{(x_3 - x_1)^2}{8(3x_1 + x_3)}$$

$$= \frac{-8(3x_1 + x_3)x_1 - (x_3 - x_1)^2}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(3x_1 + x_3)}$$

$$= \frac{-24x_1^2 - 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(3x_1 + x_3)} = \frac{-25x_1^2 - 6x_1x_3 - x_3^2}{8(3x_1 + x_3)}$$

$$= -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)} = -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

## 2.5 The case where $x_2 = -x_3$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 + x_2}{2x_1 - 4x_2 - 2x_2} = \frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 + 2x_3 + x_3}{2} = \frac{x_1 + 3x_3}{2} = -b + 2x_3$$

$$b = \frac{x_3 - x_1}{2} = \frac{4x_3 - x_1 - 3x_3}{2} = -a + 2x_3$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_2^2 - 8x_1x_2 + 8x_2x_2 + 2x_1x_2}{4}$$

$$= \frac{x_1^2 + 25x_2^2 - 6x_1x_2}{4} = \frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}$$

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}}{4\frac{x_1 - 3x_2}{2}} = -\frac{\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}}{2(x_1 - 3x_2)} = -\frac{(x_1 - 3x_2)^2 + 16x_2^2}{8(x_1 - 3x_2)}$$

$$= -\left(\frac{(x_1 - 3x_2)}{8} + \frac{2x_2^2}{(x_1 - 3x_2)}\right)$$

$$x_{ip} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 - x_2)x_2 - (-x_1 - x_2)^2}{8(x_1 - 2x_2 - x_2)}$$

$$= \frac{8(x_1 - 3x_2)x_2 - (-x_1 - x_2)^2}{8(x_1 - 3x_2)} = \frac{8x_1x_2 - 24x_2^2 - (x_1^2 + x_2^2 + 2x_1x_2)}{8(x_1 - 3x_2)}$$

$$= \frac{8x_1x_2 - 24x_2^2 - x_1^2 - x_2^2 - 2x_1x_2}{8x_1 - 24x_2} = \frac{-x_1^2 - 25x_2^2 + 6x_1x_2}{8x_1 - 24x_2}$$

$$= \frac{-x_1^2 - 25x_2^2 + 6x_1x_2}{8x_1 - 24x_2}$$

$$x_{ip} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(-x_2 - x_1)^2}{8(x_1 - 2x_2 - x_2)} = x_2 - \frac{(-x_2 - x_1)^2}{8(x_1 - 3x_2)}$$

$$= \frac{8x_2(x_1 - 3x_2) - (x_1 + x_2)^2}{8(x_1 - 3x_2)}$$

## 2.6 The case where $x_1 = -x_3$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 + x_1}{2x_1 - 4x_2 - 2x_1} = \frac{2x_1}{-4x_2} = -\frac{x_1}{2x_2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_2 - x_1}{2} = -x_2$$

$$b = \frac{x_3 - x_1}{2} = \frac{-2x_1}{2} = -x_1$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_1^2 - 8x_1x_2 + 8x_2x_1 + 2x_1x_1}{4}$$

$$= \frac{4x_1^2 + 16x_2^2}{4} = x_1^2 + 4x_2^2$$

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{x_1^2 + 4x_2^2}{4(-x_2)} = \frac{x_1^2 + 4x_2^2}{4x_2}$$

$$x_{ip} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 - x_1)x_2 - (-x_1 - x_1)^2}{8(x_1 - 2x_2 - x_1)}$$

$$= \frac{8(-2x_2)x_2 - (-2x_1)^2}{8(-2x_2)} = \frac{-16x_2^2 - 4x_1^2}{-16x_2} = \frac{4x_2^2 + x_1^2}{4x_2}$$

$$x_{ip} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(-x_1 - x_1)^2}{8(x_1 - 2x_2 - x_1)} = x_2 - \frac{4x_1^2}{-16x_2} = x_2 + \frac{x_1^2}{4x_2} = \frac{4x_2^2 + x_1^2}{4x_2}$$

# 3 Reasoning on the results

Comparing the results, we can conclude that there are only 3 types of quadratics possibilities:

- Type red: where $|a| \neq |b| = 0$ similar to the example $x = Y[y] = y^2 + 3$.
- Type green: where $|a| \neq |b| \neq 0$ similar to the example $x = Y[y] = 4y^2 - 2y + 1$.
- Type blue: where $|a| = |b| \neq 0$ similar to the example $x = Y[y] = y^2 - y$.

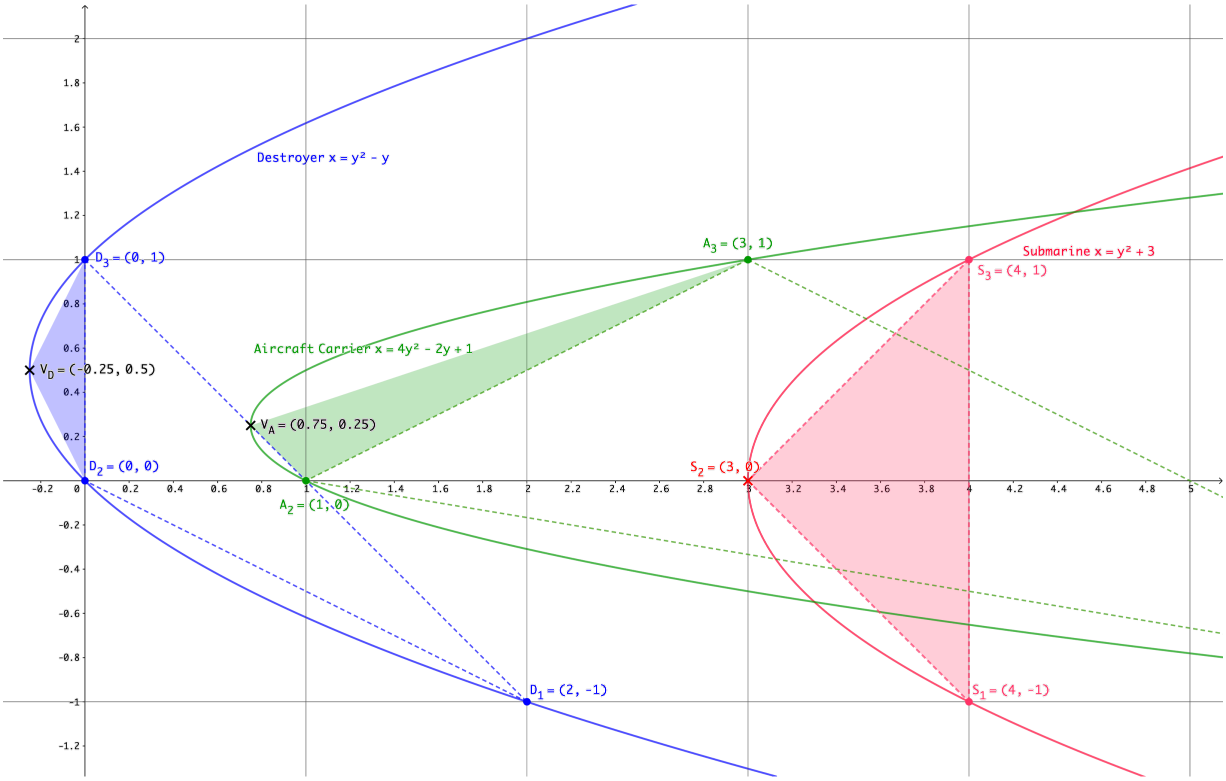


Figure 1. The 3 types possible of parabolas

Note that the hypothesis of $|b| > |a|$ is discarded. Whatever $|b| > |a|$, the offset is not zero and we will use our offset equation to get $|a| \geq |b|$ where the offset is zero.

There is no other possibility of quadratic classification besides those 3 above described. All others are equivalent to one of these cases.

Studying the 3 unique possibilities of types of quadratics we perceive that in each of them there are fundamental characteristics that distinguish one from the two others.

These 3 fundamental characteristics generate different types of Integers sequences in terms of

- Type red: duplication in all Integers but only one single Integer in the inflection point.
- Type green: no repeated Integer.
- Type blue: duplication in all Integers.

These 3 fundamental characteristics are related between the inflection point position in the XY plane and the closest 3 Integers elements to the respective inflection point.

## 3.1 Triangles in quadratics

From the inflection point, the first 3 Integers generated by these quadratics that are closer to the respective inflection point, form a triangle represented in the diagrams above with dashed lines:

- Type red: $|a| \neq |b| = 0$ exemplified in dashed red color TRIANGLE $(S_1, S_2, S_3)$.

  Note that in this Type red case, the closest Integer to the inflection point generated by this quadratic is the inflection point itself. Due to the intrinsic symmetry of the quadratic, the other 2 elements closest to the inflection point are equidistant from the inflection point. So, these two elements on an Integer sequence will be duplicated as well as all other elements generated by this quadratic. The only element not duplicated is the inflection point itself. Additionally, notice that whenever $b = 0$, then $y_{ip} = 0$ for offset zero. This means that the values of $y$ for the two other nearest inflection point elements are in $y = -1$ and $y = 1$.
- Type green: $|a| \neq |b| \neq 0$ exemplified in dashed green color TRIANGLE $(A_1, A_2, A_3)$.
- Type blue: $|a| = |b| \neq 0$ exemplified in dashed blue color TRIANGLE $(D_1, D_2, D_3)$.

# 4 Nomenclature

As a direct consequence of these properties listed above, we realize that we can use the inflection point of each quadratic $Y[y] = ay^2 + by + c$ as a reference point to "look" at the Integers numbers that each quadratic equation generates in X-axis.

We just need to define the 3 elements closest to the inflection point to determine the entire infinite quadratic sequence.

Then, we make an analogy between the 3 closest Integer elements to the inflection point with the “Battle-Ship Game” to classify any possible quadratic in one of the 3 types:

- Type red:
  - It has almost total duplication in the Integer sequence. Only the inflection point element is not duplicated.
  - It has one Integer element over the inflection point because in offset zero we have $|a| \neq |b| = 0$.
  - Using the “Battle-Ship Game” analogy, that is the reason for the name “submarine”. We will abbreviate it in our studies as “SUB”.
- Type green:
  - It has no duplication in the Integer sequence. All elements appear once.
  - It has 3 non-equidistant Integer elements closest from the inflection point because in offset zero we have $|a| > |b| \neq 0$.
  - Using the “Battle-Ship Game” analogy, that is the reason for the name “aircraft carrier”. We will abbreviate it in our studies as “ACC”.
- Type blue:
  - All elements of the Integer sequence are duplicated. All elements appear twice.
  - It has 2 equidistant Integer elements closest from the inflection point because $|a| = |b| \neq 0$.
  - Using the “Battle-Ship Game” analogy, that is the reason for the name “destroyer”. We will abbreviate it in our studies as “DES”.

Note that any parabola curve in the XY plane of the form $Y[y] = ay^2 + by + c$ will have the “opening mouth”, aperture of the parabola, towards the right or left.

- If $a > 0$, then the aperture will be facing right. This remembers the letter “C”. So, we will be classifying it in our studies as a C-type parabola or C-type quadratic.
- If $a < 0$, then the aperture will be facing left. This remembers the letter “D”. So, we will be classifying it in our studies as a D-type parabola or D-type quadratic.

# 5 Features of the "Submarine" (SUB) type quadratics

- Classification red: quadratics where $|a| \neq |b| = 0$ will be called as "submarine" quadratic with abbreviation SUB.
- In this class, for offset equal to zero, $y_{ip} = -\frac{b}{2a} = 0$.
    - For offset different from zero, $y_{ip} = -\frac{b}{2a} = n \in \mathbb{Z}, (n \neq 0)$.
- Because of the index $y_{ip}$ always will be an Integer, then the value $x_{ip}$ will be an element of the Integer sequence generated by this class of quadratic.
- For offset zero, the closest Integer index to the inflection point is only $y = 0$.
- For offset zero, the closest Integer element to the inflection point is only $Y[0] = Y[y_{ip}] = c$.
- For offset zero, $Y[0]$ will be the inflection point element. Using the "Battle-Ship Game" analogy, that is the reason for the name "submarine" (SUB).
- For any offset, any value $Y[y \neq y_{ip}]$ will be associated with two values of index y. This means almost total symmetry concerning $y_{ip}$ and any Integer sequence will have almost all its elements duplicated equidistant from the inflection point, except for element $Y[y_{ip}]$.
- For offset zero, the indexes of symmetry will be $1 \leq y < \infty$ and $-1 \geq y > -\infty$.
- For any offset, the 3 closest elements to the inflection point form a triangle $(S_1, S_2, S_3)$ isosceles.
- For any offset, the inflection point and the 2 closest elements to the inflection point form the same triangle $(S_1, S_2, S_3)$ isosceles.
- There is only one triangle.

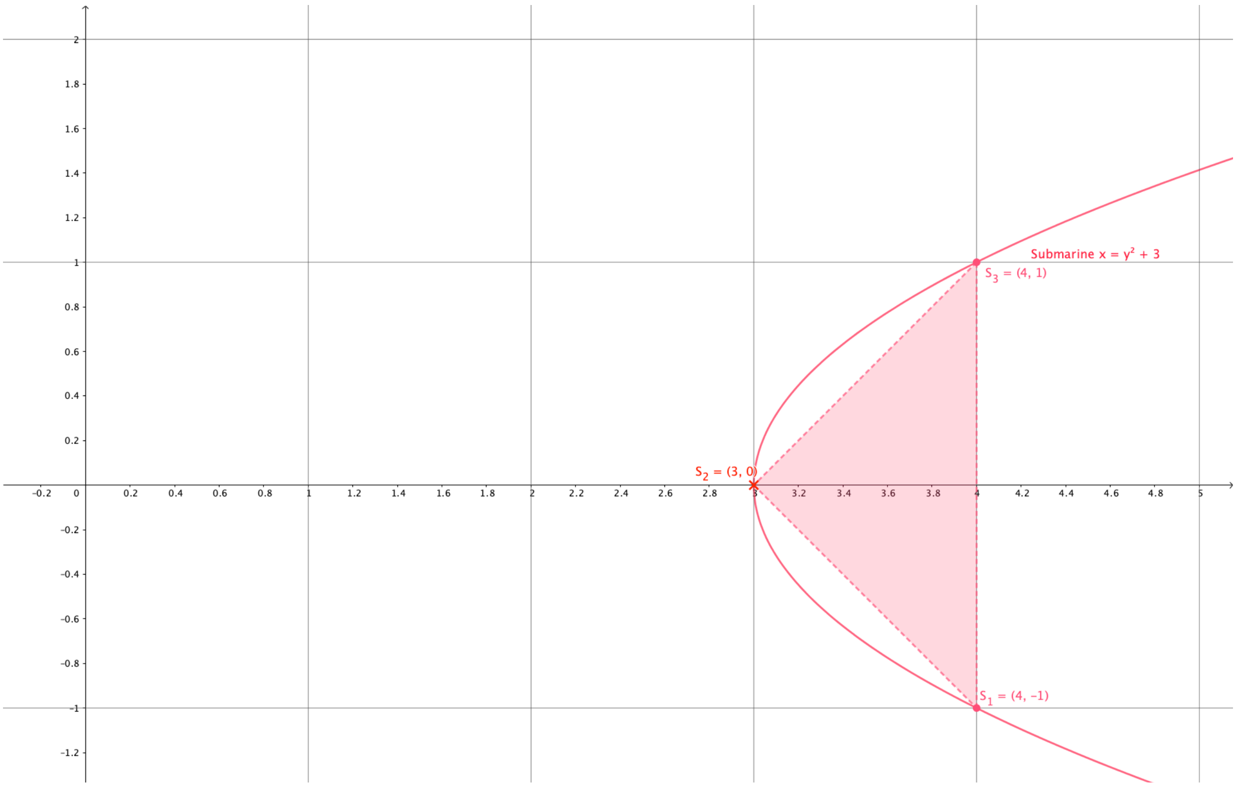


Figure 2. The submarine parabola example

# 6 Features of the "Aircraft carrier" (ACC) type quadratics

- Classification green: quadratics where $|a| \neq |b| \neq 0$ will be called as "aircraft carrier" quadratic with abbreviation ACC.
- In this class, for offset equal to zero, $0 < |y_{ip}| = \left|-\frac{b}{2a}\right| < 0.5$.
    - For offset different from zero, $n < |y_{ip}| = \left|-\frac{b}{2a}\right| < (n + 0.5)$, $(n \neq 0), n \in \mathbb{Z}$.
- Because of the index $y_{ip}$ will never be an Integer, then the value $x_{ip}$ will not be an element of the Integer sequence generated by this class of quadratic.
- For offset zero, the closest indexes to the inflection point are $y = 0$ and $y = 1$.
- For offset zero, the closest elements to the inflection point are $Y[0] = c$ and $Y[1] \neq c$.
- For offset zero, $Y[0] \neq Y[1]$ will be the 2 closest and not equidistant Integers elements to the inflection point. Using the "Battle-Ship Game" analogy, that is the reason for the name "aircraft carrier" ACC.
- For any offset, $Y[y]$ will be associated with only one value of index y. This means total asymmetry concerning $y_{ip}$ and any Integer sequence will have all its elements once.
- For offset zero, there are no indexes of symmetry.
- For any offset, the 3 closest elements to the inflection point form a triangle $(A_1, A_2, A_3)$ not isosceles.
- For any offset, the inflection point and the 2 closest elements to the inflection point form a triangle $(V_A, A_2, A_3)$ not isosceles.
- Triangle $(A_1, A_2, A_3)$ is always different from triangle $(V_A, A_2, A_3)$.

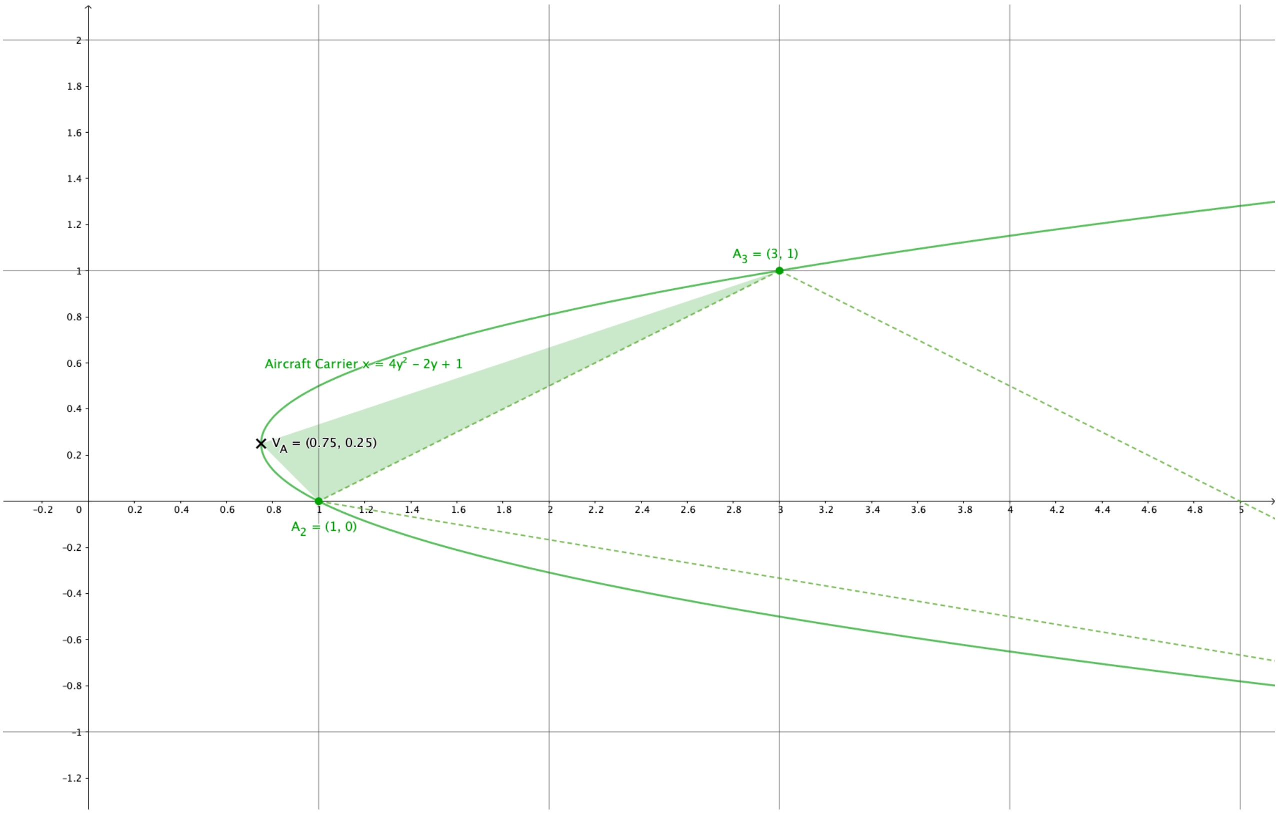


Figure 3. The aircraft-carrier parabola example

# 7 Features of the "Destroyer" (DES) type quadratics

- Classification blue: quadratics where $|a| = |b| \neq 0$ will be called as "destroyer" quadratic with abbreviation DES.
- In this class, for offset equal to zero, $y_{ip} = -\frac{b}{2a} = 0.5$.
  - For offset different from zero, $y_{ip} = -\frac{b}{2a} = (n + 0.5), (n \neq 0), n \in \mathbb{Z}$.
- Because of the index $y_{ip}$ never will be an Integer, then the value $x_{ip}$ will not be an element of the Integer sequence generated by this class of quadratic.
- For offset zero, the closest indexes to the inflection point are $y = 0$ and $y = 1$.
- For offset zero, the closest elements to the inflection point are $Y[0] = Y[1] = c$.
- For offset zero, $Y[0] = Y[1]$ will be the 2 closest and equidistant Integers elements to the inflection point. Using the "Battle-Ship Game" analogy, that is the reason for the name "destroyer" (DES).
- For any offset, any value $Y[y]$ will be associated with two values of index y. This means total symmetry of the Integers concerning $y_{ip}$. Any Integer sequence will have all its elements duplicated equidistant from the inflection point.
- For offset zero, the indexes of symmetry will be $1 \leq y < \infty$ and $0 \geq y > -\infty$.
- For any offset, the 3 closest elements to the inflection point form a triangle $(D_1, D_2, D_3)$ not isosceles.
- For any offset, the inflection point and the 2 closest elements to the inflection point form a triangle $(V_D, D_2, D_3)$ isosceles.
- Triangle $(D_1, D_2, D_3)$ is different from triangle $(V_D, D_2, D_3)$.

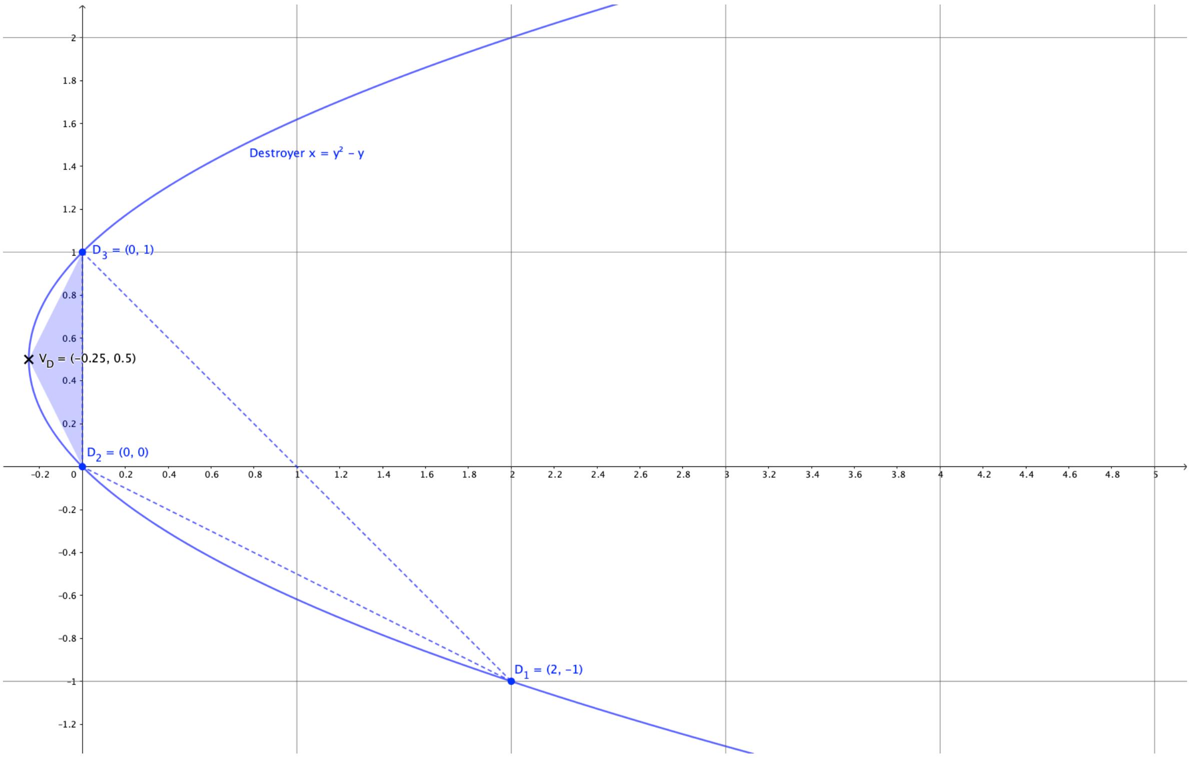


Figure 4. The destroyer parabola example

# 8 Summary

| | | | | | | |
|---|---|---|---|---|---|---|
| | Representation with the coefficients | $a$ | $b$ | $-\frac{b}{2a}$ | $-\frac{\Delta}{4a} = -\frac{b^2-4ac}{4a}$ | $\Delta = b^2 - 4ac$ |
| | Equations with the 3 consecutive elements | $\frac{x_1 - 2x_2 + x_3}{2}$ | $\frac{x_3 - x_1}{2}$ | $\frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3}$ | $x_2 - \frac{(-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)}$ | $\frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}$ |
| Case | Classification | Coef. a | Coef. b | $y_{ip}$ | $x_{ip}$ | Discriminant $= \Delta$ |
| $\boldsymbol{x_1 = x_2}$ | *DES* $\lvert a\rvert = \lvert b\rvert \neq 0$ | $\frac{x_3 - x_1}{2} = b$ | $\frac{x_3 - x_1}{2} = a$ | $-\frac{1}{2}$ | $\frac{9x_1 - x_3}{8}$ | $(-x_1 + x_3)\frac{-9x_1 + x_3}{4}$ |
| $\boldsymbol{x_2 = x_3}$ | *DES* $\lvert a\rvert = \lvert b\rvert \neq 0$ | $\frac{x_1 - x_2}{2} = -b$ | $\frac{x_3 - x_1}{2} = -a$ | $\frac{1}{2}$ | $\frac{-x_1 + 9x_2}{8}$ | $(x_1 - x_2)\frac{x_1 - 9x_2}{4}$ |
| $\boldsymbol{x_1 = x_3}$ | *SUB* $\lvert a\rvert > \lvert b\rvert = 0$ | $x_1 - x_2$ | $0$ | $0$ | $x_2$ | $4x_2(-x_1 + x_2)$ |
| $\boldsymbol{x_1 = -x_2}$ | *ACC* $\lvert a\rvert \neq \lvert b\rvert \neq 0$ | $\frac{3x_1 + x_3}{2} = b + 2x_1$ | $\frac{x_3 - x_1}{2} = a - 2x_1$ | $\frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$ | $-\left(\frac{3x_1 + x_3}{8} + \frac{2x_1^2}{3x_1 + x_3}\right)$ | $\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}$ |
| $\boldsymbol{x_2 = -x_3}$ | *ACC* $\lvert a\rvert \neq \lvert b\rvert \neq 0$ | $\frac{x_1 + 3x_3}{2} = -b + 2x_3$ | $\frac{x_3 - x_1}{2} = -a + 2x_3$ | $\frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$ | $-\left(\frac{x_1 - 3x_2}{8} + \frac{2x_2^2}{x_1 - 3x_2}\right)$ | $\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}$ |
| $\boldsymbol{x_1 = -x_3}$ | *ACC* $\lvert a\rvert \neq \lvert b\rvert \neq 0$ | $-x_2$ | $-x_1$ | $-\frac{x_1}{2x_2}$ | $\frac{x_1^2 + 4x_2^2}{4x_2}$ | $x_1^2 + 4x_2^2$ |

Table 1. Parameters summary

| Classification of Quadratics of the form $Y[y] = x = ay^2 + by + c$ | | | |
|---|---|---|---|
| Classification Name | SUBMARINE (SUB) | AIRCRAFT CARRIER (ACC) | DESTROYER (DES) |
| Fundamental Characteristic I | They are the quadratics that possess infinitely many duplicated Integers less one. | They are the quadratics that possess no duplicated Integer. | They are the quadratics that possess all its Integers duplicated. |
| Fundamental Characteristic II | They are the parabolas that possess one Integer number at the inflection point and infinitely many duplicated Integers equidistant to the inflection point. | They are the parabolas that there are no two Integers equidistant to the inflection point. | They are the parabolas that possess all its infinitely many duplicated Integers equidistant to the inflection point. |
| Math definition when the offset is zero ($f = 0$) | $\|a\| \neq \|b\| = 0$ | $\|a\| \neq \|b\| \neq 0$ | $\|a\| = \|b\| \neq 0$ |
| $y_{ip} = -\dfrac{b}{2a}$ | $y_{ip} \in \mathbb{Z}$ | $y_{ip} \neq \left(\frac{Odd}{2}\right)$ *and* $y_{ip} \notin \mathbb{Z}$ | $y_{ip} = \left(\dfrac{Odd}{2}\right)$ |
| $x_{ip} = Y[y_{ip}] = -\frac{\Delta}{4a}$ an $\mathbb{Z}$ element sequence? | *yes*, $Y[y_{ip}] = c \in \mathbb{Z}$ | *no*, $Y[y_{ip}] = c - \frac{b^2}{4a} \notin \mathbb{Z}$ | *no*, $Y[y_{ip}] = c - \frac{a}{4} \notin \mathbb{Z}$ |
| Closest elements to the inflection point | $Y[y_{ip} - 1] = Y[y_{ip} + 1]$ | $Y[y_{ip}] \neq Y[y_{ip} \pm 1]$ | $Y[y_{ip} - 0.5] = Y[y_{ip} + 0.5)$ |
| Symmetry when the offset is zero ($f = 0$) | $Y[y] = Y[-y]$ | *no symmetry* | $Y[y] = Y[-y + 1]$ |
| Symmetry in any offset $f = -h$ | $Y[y + h] = Y[-y + h]$ | *no symmetry* | $Y[y + h] = Y[-y + 1 + h]$ |
| Indexes for symmetry when the offset is zero ($f = 0$) | $1 \leq y < \infty$ *and* $-1 \geq y > -\infty$ | *no symmetry* | $1 \leq y < \infty$ *and* $0 \geq y > -\infty$ |
| Triangle formed by the ip and the 2 closest Integers | *Isosceles* | *never Isosceles* | *Isosceles* |
| Triangle formed by the first three Integers elements | *Isosceles* | *never Isosceles* | *never Isosceles* |
| Offset ($f$) equation | $f = roundz(y_{ip})$ | $f = roundz(y_{ip})$ | $f = roundz(y_{ip})$ |
| Mouth aperture of the parabola | *For* $a > 0$ *we call as C-type.*<br>*For* $a < 0$ *we call as D-type.* | *For* $a > 0$ *we call as C-type.*<br>*For* $a < 0$ *we call as D-type.* | *For* $a > 0$ *we call as C-type.*<br>*For* $a < 0$ *we call as D-type.* |

Table 2. Classification summary

# Acknowledgments

I would like to thank all the essential support and inspiration provided by Mr. H. Bli Shem and my Family.

# References

[1] *The On-Line Encyclopedia of Integer Sequences*, available online at http://oeis.org.

[2] *Offset in Quadratics*