Coefficients of Higher Powers of a Matrix: Companion Matrix

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COEFFICIENTS OF HIGHER POWERS OF A MATRIX: COMPANION MATRIX

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ABSTRACT. For any NxN square matrix, X; { X^j; for j≥N}, can be expressed in terms of $I, X, X^2, X^3, ...X^{N-1}$. The coefficients of them are proved to be expressible in terms of companion matrix of X.

1. INTRODUCTION

Linear Algebra, as a branch of mathematics, arose in an effort to solve linear System of equations. Linear algebraic concepts such as Rank, Column/ Row Space, Eigenvalues, Eigenvectors, etc. are introduced and utilized successfully. Various interesting theorems were proved. One interesting and important result was the Cayley Hamilton theorem.

In queuing theory and other research areas, matrix Polynomial equations naturally arise. Such matrix polynomial equations in which the degree of unknown matrix is larger than the dimension of matrix find many applications. In view of Cayley - Hamilton theorem, the powers of the unknown matrix X, equal to and higher than the dimension of a matrix can be expressed in terms of $I, X, X^2, X^3, ...X^{N-1}$, where N is the dimension of the unknown matrix X. Intuitively, it is clear that the coefficients of any power of X, can be expressed in the terms of coefficients of the characteristic polynomial of X(by the utilization of the Cayley Hamilton theorem). This research paper addresses and solves the question of determining coefficients of $I, X, X^2, X^3, ...X^{N-1}$ leading to any arbitrary power 'k' of X i.e.

$$X^K = (b_0^{(K)} * I) + (b_1^{(K)} * X) + (b_2^{(K)} * X^2) + (b_3^{(K)} * X^3)... + (b_{N-1}^{(K)} * X^{N-1}).$$

Determination of $b_i^{(K)}$(where $i \in \{1, 2, 3, ..., N-1\}$) in terms of coefficients of polynomial Det(λ*I-X) is provided.

This research paper is organized as follows: In Section 2, the related research literature is reviewed. In Section 3, an expression for higher powers of a matrix i.e.,X^j; for j≥N in terms of $I, X, X^2, X^3, ...X^{N-1}$ is derived using the companion matrix associated with the characteristic polynomial of X. In Section 4, the conclusion is reported.

2. Review of Related Research Literature

Cayley, Hamilton proved a theorem which states that every square matrix satisfies its characteristic polynomial equation[1][2]. Linear Algebraists also realized that using such a theorem, any matrix power series can be expressed as a Matrix Polynomial. This result indirectly leads to the conclusion that a matrix polynomial equation(with coefficient matrices, unknown matrix X being NxN matrices) can be expressed in terms of $I, X, X^2, X^3, ...X^{N-1}$. To the best of our knowledge, it is not known what are the coefficients of $I, X, X^2, X^3, ...X^{N-1}$ to express X^j, for an arbitrary value j≥N. More interestingly, how to express a matrix power series equation as a matrix polynomial equation is also unknown(i.e., Coefficients of $I, X, X^2, X^3, ...X^{N-1}$ are not known in literature).

3. Companion Matrix of X: Higher Power

For the purpose of clarity, we first consider the case where X is a 2x2 matrix.
When X is a 2x2 matrix, using the Cayley Hamilton theorem, we infer that X^j's for j≥2(with fixed pair of eigenvalues of X) can be expressed in terms of matrices X,I using scalar coefficients determined by the coefficients of characteristic polynomial of X. For instance, we readily have that

$$X^3 = (b_1^2 - b_0)X + (b_1 b_0)I$$

where $Det(\lambda$*I-X)= $\lambda^2 + b_1\lambda + b_0$, with $b_1 = -Trace(X)$ and $b_0 = Determinant(X)$ $(X^2 I + b_1 X + b_0 I \equiv \overline{O})$
Now letting,$b_1^{(2)} = b_1$,$b_0^{(2)} = b_0$ and $b_1^{(3)} = (b_1^2 - b_0)$, $b_0^{(3)} = b_1 b_0$.
We have that following recursive equation holds:

$$\begin{pmatrix} -b_1^{(3)} \\ -b_0^{(3)} \end{pmatrix} = \begin{pmatrix} -b_1^{(2)} & 1 \\ -b_0^{(2)} & 0 \end{pmatrix} * \begin{pmatrix} -b_1^{(2)} \\ -b_0^{(2)} \end{pmatrix}$$

$$= C_{(2)} \begin{pmatrix} -b_1^{(2)} \\ -b_0^{(2)} \end{pmatrix}$$

where C_2 is a Companion Matrix. This recursion was first observed in [3]. By letting, $X^4 = -b_1^{(4)} X - b_0^{(4)} I$.
It can be readily shown that

$$\begin{pmatrix} -b_1^{(4)} \\ -b_0^{(4)} \end{pmatrix} = (C_{(2)})^2 \begin{pmatrix} -b_1^{(2)} \\ -b_0^{(2)} \end{pmatrix}$$

In general, X^M can be expressed in the terms of (X,I) with two suitable coefficients. The two coefficients are obtained using the following recursion(generalization of the above result):

$$\begin{pmatrix} -b_1^{(M)} \\ -b_0^{(M)} \end{pmatrix} = (C_{(2)})^{M-2} \begin{pmatrix} -b_1^{(2)} \\ -b_0^{(2)} \end{pmatrix}$$

Thus, coefficients of higher powers of X can be expressed in terms of a 2x2 companion matrix and the coefficients of characteristic polynomial of X. Now we consider the case where B/X is an NxN matrix.

In the following, we consider B instead of X. In general, B^M can be expressed in the terms of $I, B, B^2, B^3, ...B^{N-1}$ with suitable coefficients. The coefficients can be obtained using the following recursion:

$$\begin{pmatrix} -b_{N-1}^{(m)} \\ -b_{N-2}^{(m)} \\ -b_{N-3}^{(m)} \\ . \\ . \\ . \\ -b_2^{(m)} \\ -b_1^{(m)} \\ -b_0^{(m)} \end{pmatrix} = (C_{NxN})^{m-N} \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}$$

where $B_{NxN} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & . & . & . & . & a_{1N-1} & a_{1N} \\ a_{21} & a_{22} & a_{23} & a_{24} & . & . & . & . & a_{2N-1} & a_{2N} \\ a_{31} & a_{32} & a_{33} & a_{34} & . & . & . & . & a_{3N-1} & a_{3N} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ a_{N-11} & a_{N-12} & a_{N-13} & a_{N-14} & . & . & . & . & a_{N-1N-1} & a_{N-1N} \\ a_{N1} & a_{N2} & a_{N3} & a_{N4} & . & . & . & . & a_{NN-1} & a_{NN} \end{pmatrix}$

and $C_{NxN} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}$

i.e. C is the Companion Matrix associated with Characteristic polynomial of X.

Theorem: $(B)^m$ = -$b_{N-1}^{(m)}*B^{N-1}$ - $b_{N-2}^{(m)}*B^{N-2}$ - $b_{N-3}^{(m)}*B^{N-3}$ . . . . $b_1^{(m)}$*B - $b_0^{(m)}$*I, where B=NxN matrix and $m \geq$ N $\geq$ 3.

Proof. For the purpose of clarity, we now write down the cases $m = N$, $m = N + 1$, $m = N + 2$.

Base Case : $m = N$

$$\begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{N-N} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}^{0}_{NxN} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

Case 2 $: m = N + 1$

$$\begin{pmatrix} -b_{N-1}^{(n+1)} \\ -b_{N-2}^{(n+1)} \\ -b_{N-3}^{(n+1)} \\ . \\ . \\ . \\ -b_2^{(n+1)} \\ -b_1^{(n+1)} \\ -b_0^{(n+1)} \end{pmatrix}_{Nx1} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}^{N-N+1}_{NxN} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}^{1}_{NxN} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} b_{N-1}^2 - b_{N-2} \\ b_{N-2}b_{N-1} - b_{N-3} \\ b_{N-3}b_{N-1} - b_{N-4} \\ b_{N-4}b_{N-1} - b_{N-5} \\ . \\ . \\ . \\ . \\ b_2 b_{N-1} - b_1 \\ b_1 b_{N-1} - b_0 \\ b_0 b_{N-1} \end{pmatrix}_{Nx1}$$

Case 3 $: m = N + 2$

$$\begin{pmatrix} -b_{N-1}^{(n+2)} \\ -b_{N-2}^{(n+2)} \\ -b_{N-3}^{(n+2)} \\ . \\ . \\ . \\ -b_2^{(n+2)} \\ -b_1^{(n+2)} \\ -b_0^{(n+2)} \end{pmatrix}_{Nx1} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}^{N-N+2}_{NxN} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}^2_{NxN} * \begin{pmatrix} -b^{(N)}_{N-1} \\ -b^{(N)}_{N-2} \\ -b^{(N)}_{N-3} \\ . \\ . \\ . \\ -b^{(N)}_2 \\ -b^{(N)}_1 \\ -b^{(N)}_0 \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}^1_{NxN} * \begin{pmatrix} b^2_{N-1} - b_{N-2} \\ b_{N-2}b_{N-1} - b_{N-3} \\ b_{N-3}b_{N-1} - b_{N-4} \\ b_{N-4}b_{N-1} - b_{N-5} \\ . \\ . \\ . \\ . \\ b_2b_{N-1} - b_1 \\ b_1b_{N-1} - b_0 \\ b_0b_{N-1} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1}b^2_{N-1} + b_{N-1}b_{N-2} + b_{N-2}b_{N-1} - b_{N-3} \\ -b_{N-2}b^2_{N-1} + b_{N-2}b_{N-2} + b_{N-3}b_{N-1} - b_{N-3} \\ -b_{N-3}b^2_{N-1} + b_{N-3}b_{N-2} + b_{N-4}b_{N-1} - b_{N-3} \\ -b_{N-4}b^2_{N-1} + b_{N-4}b_{N-2} + b_{N-5}b_{N-1} - b_{N-3} \\ . \\ . \\ . \\ . \\ -b_2b^2_{N-1} + b_2b_{N-2} + b_1b_{N-1} - b_0 \\ -b_1b_{N-1} + b_1b_{N-2} + b_0b_{N-1} \\ -b_0b_{N-1} + b_0b_{N-2} \end{pmatrix}_{Nx1}$$

Now, let the induction hypothesis hold for m= N+K. So, we have

$$\begin{pmatrix} -b_{N-1}^{(N+K)} \\ -b_{N-2}^{(N+K)} \\ -b_{N-3}^{(N+K)} \\ . \\ . \\ . \\ -b_{2}^{(N+K)} \\ -b_{1}^{(N+K)} \\ -b_{0}^{(N+K)} \end{pmatrix}_{Nx1} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{N-N+K} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_{2}^{(N)} \\ -b_{1}^{(N)} \\ -b_{0}^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{K} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_{2}^{(N)} \\ -b_{1}^{(N)} \\ -b_{0}^{(N)} \end{pmatrix}_{Nx1}$$

$$= -B^{N+K}.$$

Inductive Step $(m \longrightarrow N+K+1)$:
we have to prove the below equation.

$$\begin{pmatrix} -b_{N-1}^{(N+K+1)} \\ -b_{N-2}^{(N+K+1)} \\ -b_{N-3}^{(N+K+1)} \\ . \\ . \\ . \\ -b_{2}^{(N+K+1)} \\ -b_{1}^{(N+K+1)} \\ -b_{0}^{(N+K+1)} \end{pmatrix}_{Nx1} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{K+1} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_{2}^{(N)} \\ -b_{1}^{(N)} \\ -b_{0}^{(N)} \end{pmatrix}_{Nx1}$$

Proof. LHS:

$$\begin{pmatrix} -b_{N-1}^{(N+K+1)} \\ -b_{N-2}^{(N+K+1)} \\ -b_{N-3}^{(N+K+1)} \\ . \\ . \\ . \\ -b_{2}^{(N+K+1)} \\ -b_{1}^{(N+K+1)} \\ -b_{0}^{(N+K+1)} \end{pmatrix}_{Nx1} = \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{N-N+K+1} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_{2}^{(N)} \\ -b_{1}^{(N)} \\ -b_{0}^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN} * (-B^{N+K})$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{1} * \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{K} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_{2}^{(N)} \\ -b_{1}^{(N)} \\ -b_{0}^{(N)} \end{pmatrix}_{Nx1}$$

$$= \begin{pmatrix} -b_{N-1} & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-2} & 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ -b_{N-3} & 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b_1 & 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ -b_0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}_{NxN}^{K+1} * \begin{pmatrix} -b_{N-1}^{(N)} \\ -b_{N-2}^{(N)} \\ -b_{N-3}^{(N)} \\ . \\ . \\ . \\ -b_2^{(N)} \\ -b_1^{(N)} \\ -b_0^{(N)} \end{pmatrix}_{Nx1}$$

=RHS

□

Thus, LHS= RHS

Hence proved. □

4. Conclusion

In this research paper, it is proved that using the Companion Matrix associated with characteristic polynomial of X, the coefficients of $I, X, X^2, X^3, ...X^{N-1}$ that express $\{X^j$; for j≥N$\}$ in the form of

$$X^j = (b_0^{(j)} * I) + (b_1^{(j)} * X) + (b_2^{(j)} * X^2) + (b_3^{(j)} * X^3)... + (b_{N-1}^{(j)} * X^{N-1})$$

can be explicitly determined. This result has important consequence for matrix valued analytic functions.

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