On Solé and Planat Criterion for the Riemann Hypothesis

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Abstract

The Riemann hypothesis is the assertion that all non-trivial zeros have real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{q \leq q_n} (1 + \frac{1}{q}) > e^{\gamma} \cdot \log \theta(q_n)$ holds for all prime numbers $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and log is the natural logarithm. In this note, using Solé and Planat criterion, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Riemann zeta function, Prime numbers, Chebyshev function

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1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex

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numbers with real part $\frac{1}{2}$. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems.

In number theory, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm.

Proposition 1.1. *There are infinitely many natural numbers x such that [1, pp. 1]:*

$$\theta(x) > x.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 1.2. *It is known that[2, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number).

Franz Mertens obtained some important results about the constants B and H (1874). We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3, (17.) pp. 54].

Proposition 1.3. *We have [4, Lemma 2.1 (1) pp. 359]:*

$$\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right)-\frac{1}{q_k}\right)=\gamma-B=H.$$

On the sum of the reciprocals of all prime numbers not exceeding x, we have

Proposition 1.4. *For $x \geq 2$ [4, Lemma 2.1 (1) pp. 359]:*

$$\sum_{q\leq x}\frac{1}{q}=B+\log\log(x)+O\left(\frac{1}{\log x}\right).$$

In mathematics, $\Psi(n)=n\cdot\prod_{q\mid n}\left(1+\frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)>\frac{e^{\gamma}}{\zeta(2)}\cdot\log\theta(q_n).$$

Next, we have Solé and Planat Theorem:

Proposition 1.5. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers $q_n > 3$ if and only if the Riemann hypothesis is true [5, Theorem 4.2 pp. 5].*

A natural number N_k is called a primorial number of order k precisely when,

$$N_k=\prod_{i=1}^{k}q_i.$$

We define $R(n)=\frac{\Psi(n)}{n\cdot\log\log n}$ for $n\geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n)>\frac{e^{\gamma}}{\zeta(2)}$ is satisfied. There are several statements out from the Riemann hypothesis assumption:

Proposition 1.6. *We have [5, Proposition 3. pp. 3]:*

$$\lim_{k\to\infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

Putting all together yields a proof for the Riemann hypothesis using the Chebyshev function.

2. Central Lemma

This is a key Lemma.

Lemma 2.1.

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - \log\left(1+\frac{1}{q_k}\right)\right) = \log(\zeta(2)) - H.$$

Proof. We obtain that

$$\begin{aligned}
\log(\zeta(2)) - H &= \log(\prod_{k=1}^{\infty}\frac{q_k^2}{q_k^2-1}) - H \\
&= \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k^2}{(q_k^2-1)}\right)\right) - H \\
&= \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k^2}{(q_k-1)\cdot(q_k+1)}\right)\right) - H \\
&= \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right) + \log\left(\frac{q_k}{q_k+1}\right)\right) - H \\
&= \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right) - \log\left(\frac{q_k+1}{q_k}\right)\right) - H \\
&= \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right) - \log\left(1+\frac{1}{q_k}\right)\right) - \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right) - \frac{1}{q_k}\right) \\
&= \sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right) - \log\left(1+\frac{1}{q_k}\right) - \log\left(\frac{q_k}{q_k-1}\right) + \frac{1}{q_k}\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{1}{q_k} - \log\left(1+\frac{1}{q_k}\right)\right)
\end{aligned}$$

by Propositions 1.2 and 1.3. $\square$

3. What if the Riemann hypothesis were false?

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Lemma 3.1. *If the Riemann hypothesis is false, then there are infinitely many prime numbers q_n for which $\mathsf{Dedekind}(q_n)$ fails (i.e. $\mathsf{Dedekind}(q_n)$ does not hold).*

Proof. The Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [5, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [6, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [6, Theorem 3 (c) pp. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

4. Main Insight

This is the main insight.

Theorem 4.1. *The Riemann hypothesis is true when for every large enough prime number $q_n > 3$, there exists another prime $q_{n'} > q_n$ such that*

$$R(N_{n'}) \leq R(N_n).$$

Proof. If the Riemann hypothesis is false and the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for every large enough prime number $q_n > 3$, then there is an infinite subsequence of natural numbers n_i such that

$$R(N_{n_{i+1}}) \leq R(N_{n_i}),$$

$q_{n_{i+1}} > q_{n_i}$ and $\mathsf{Dedekind}(q_{n_i})$ fails by Lemma 3.1.

This is a contradiction with the fact that

$$\liminf_{k\to\infty} R(N_k) = \lim_{k\to\infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}$$

by Proposition 1.6. By definition of the limit inferior for any positive real number ε, only a finite number of elements of the sequence $R(N_k)$ are less than $\frac{e^{\gamma}}{\zeta(2)} - \varepsilon$. This is a contradiction with the previous infinite subsequence and thus, the Riemann hypothesis must be true. $\square$

5. Main Theorem

This is the main theorem.

Theorem 5.1. *The Riemann hypothesis is true.*

Proof. The Riemann hypothesis is true when

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for large enough prime numbers $q_{n'} > q_n$ because of the Theorem 4.1. That is the same as

$$\frac{\prod_{q\leq q_{n'}}\left(1+\frac{1}{q}\right)}{\log\theta(q_{n'})} \leq \frac{\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)}{\log\theta(q_n)}$$

which is

$$\log\log\theta(q_{n'}) - \log\log\theta(q_n) \geq \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right)$$

after making a simple distribution in the inequality.

For every large enough prime number $q_n > 3$, there exists another prime $q_{n'} > q_n$ such that

$$\begin{aligned}
&\log\log\theta(q_{n'}) - \log\log\theta(q_n) \\
&= \log\log\theta(x) - \log\log\theta(q_n) \\
&> \log\log(x) - \log\log\theta(q_n) \\
&= B + \log\log(x) - B - \log\log\theta(q_n) \\
&= \left(\sum_{q \leq q_{n'}} \frac{1}{q}\right) + O\left(\frac{1}{\log x}\right) - B - \log\log\theta(q_n) \\
&= \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right) + O\left(\frac{1}{\log x}\right) + \alpha + \left(\sum_{q \leq q_n} \log\left(1 + \frac{1}{q}\right)\right) - \log\log\theta(q_n) - B \\
&\geq \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right)
\end{aligned}$$

where

$$\theta(x) > x$$

and

$$B + \log\log(x) = \sum_{q \leq x} \frac{1}{q} + O\left(\frac{1}{\log x}\right) = \sum_{q \leq q_{n'}} \frac{1}{q} + O\left(\frac{1}{\log x}\right)$$

could be true by Propositions 1.1 and 1.4,

$$\alpha = \sum_{q \leq q_{n'}} \left(\frac{1}{q} - \log\left(1 + \frac{1}{q}\right)\right) \gtrapprox \log(\zeta(2)) - H$$

and so,

$$
\begin{aligned}
&\alpha + \left(\sum_{q \leq q_n} \log \left(1 + \frac{1}{q} \right) \right) - \log \log \theta(q_n) - B \\
&\lessapprox \log(\zeta(2)) + \left(\sum_{q \leq q_n} \log \left(1 + \frac{1}{q} \right) \right) - \log \log \theta(q_n) - \gamma \\
&= \log \left(R(N_n) \cdot \frac{\zeta(2)}{e^{\gamma}} \right) \\
&\approx 0
\end{aligned}
$$

for large enough pair of prime numbers $(q_n, q_{n'})$ and $q_{n'} > q_n$ by Lemma 2.1 and Proposition 1.6 when

$$O\left(\frac{1}{\log x} \right) + \log \left(R(N_n) \cdot \frac{\zeta(2)}{e^{\gamma}} \right) \gtrapprox 0.$$

Consequently, the inequality

$$R(N_{n'}) \leq R(N_n)$$

holds for sufficiently large prime numbers $q_{n'} > q_n$ and therefore, the Riemann hypothesis is true. $\square$

6. Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. In general, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics.

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