



## Game Theory, the Science of Strategy

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May 12, 2022

# Game theory, the science of strategy

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## Abstract

This paper makes a simple presentation of strategy games emphasizing their application to management in general. The language used is very straightforward and mathematical symbols are avoided. Mathematical reasoning is presented descriptively. The dominant perspective is critical in that game theory has promised so much but to a certain extent has failed to fulfil its promises, mainly in real world applications. However, recent developments envisage a brighter future for game theory in both from a practical and theoretical aspect.

**Keywords:** Game theory, strategy, minimax theorem, Nash theorem

## 1. Introduction

Defined in their broadest generality, the games can abundantly be found in real life situations. International politics, the economy, family life, election campaigns and many other situations are cases in which a player seeks a strategy that results so as to obtain a certain goal in opposition to other players who are also trying to optimize their perspective. The final result depends on the set of strategies adopted by all participants.

We may say that a game is a situation in which two or more participants, the players, confront each other in order to achieve certain goals which sometimes may not be achieved simultaneously. Thus, a game is a description of strategies that include restrictions on the actions that players can take and on the players' interests in not specifying what actions the players should take. In a game each player's interests are confronted, forcing each one to develop action strategies to maximize gains or minimize losses.

As games are disputed between competitors where the result of the decisions of a player's decision depends on the actions of other players, apart from knowledge of the dispute, it is important to knowing the competitor. That is, to know how competitors may choose their strategies, their action strategy, what their interests are, what their objectives are. It is important to have information not only for the player himself, but mainly regarding the information held by the competitor.

Game theory is a discipline that seeks to understand phenomena that are observed when interacting decisions are taken. The basic premise of this theory is the rationality of decisions, i.e. it starts

from the principle that decision makers are rational and act strategically, taking into account their knowledge or expectations about the behaviour of other decision makers.

Despite the fact that theoretical ideas of the game are not entirely mathematical, game theory uses mathematics to express its ideas formally because it is thus easier to define concepts rigorously, creating independence of mathematical interests, checking the consistency of ideas and exploring the implications of the results. Consequently, the concepts and results are accurate, interposed with motivations and interpretations of concepts.

## 2. Minimax Theorem

In 1985 the Babylonian Talmud – a compilation of ancient laws and traditions which formed the basis for the Jewish religion and civil and criminal law for the first five centuries BCE was recognized as having anticipated modern game theory. Nevertheless, it was in the 1940s that it emerged with most work directed at a special class of games: **zero-sum games**. These are games in which each player gets exactly what the other loses, regardless of the possible strategy.

Von Neumann presented one of the greatest results for the constant-sum games – games where sum of the gain and loss of players is a constant (not necessarily zero) and can always be reduced to zero-sum games. He showed rigorously that there is always a rational course of action in two-player games, as long as their interests are completely opposed. Von Neumann singularly and unequivocally answered the question, **“how can I maximize my payoffs in zero-sum games with two players?”**

### 2.1. SUMUS versus SUNEK

To illustrate this result consider a duopoly example based on R. A. McCain of two companies that sell bottled juices. To facilitate the study let us call these companies SUMUS and SUNEK. Each company has a fixed cost of 5,000 monetary units (m u) regardless of the number of bottles sold. Both companies compete for the same market and have to choose between selling each bottle for the price of 1 or 2 m u. The assumptions of the problem are as follows:

- for the price of 2 u m per bottle, 5 000 bottles can be sold with a return of 10 000 m u;
- for the price of 1 u m per bottle, 10 000 bottles can be sold with a return of 10 000 m u;
- if both companies place the bottles on the market for the same price, sales will be equally divided;
- if one company places the higher price, the company with the lower price sells the entire amount, whereas the company which places the higher price does not sell anything;
- the payoffs are the profits after deducting fixed costs.

Of course, when making decisions, companies have diametrically opposed interests. What is good for SUMUS is bad for SUNEK and vice versa. Clearly, it follows that both companies will take decisions that may be classified as risk-averse, i.e., decisions that renounce some possible gains to avoid incurring unnecessary losses.

To get an overview of the situation, consider the payoff matrix that defines the normal form of the game:

		SUNEK	
		1 m u	2 m u
SUMUS	1 m u	(0,0)	(5 000,-5 000)
	2 m u	(-5 000, 5 000)	(0,0)

Figure 2.1. The "SUMUS versus SUNEK" game –normal form

Interpretation of the payoff matrix is as follows: the rows of the matrix represent the SUMUS's options and the columns represent SUNEK's options. Each ordered pair represents the earnings of each company depending on the chosen strategies. The value on the left is the gain for SUMUS, and the one on the right is the gain for SUNEK. Because it is a zero-sum game, SUMUS's earnings are symmetrical with regards to SUNEK's.

Let us begin by analysing the result from SUMUS's point of view, assuming that despite being able to reasonably predict the payoff matrix, neither company knows the strategy their competitor will adopt. Unaware of SUNEK's plans, SUMUS may proceed as follows:

- Determine the lowest payoff they can receive in each of their strategies – the minimum of each row of the payoff matrix;
- Choose the strategy that has the highest minimum - choose the line of the payoff matrix.

By doing so, SUMUS can ensure that, whatever its competitor's decision, they will not get the worst possible outcome, avoiding the less favourable results (lower minimum lines). Likewise, the company will also never achieve the best possible outcome as they ignored the best results on purpose. Applying this procedure to Figure 2.1, SUMUS obtains:

		SUNEK		min:	maximin:
		1 m u	2 m u		
SUMUS	1 m u	0	5 000	0	0
	2 m u	-5 000	0	-5 000	

Figure 2.2. The "SUMUS versus SUNEK " game – Maxmin strategy

Examining the game's results from SUNEK's point of view, adopting the same criteria, the company will seek to maximize the set of minimums in the columns of their payoff matrix, obtaining:

		<i>SUNEK</i>	
		1 m u	2 m u
<i>SUMUS</i>	1 m u	0	-5 000
	2 m u	5 000	0
		min: 0	min: -5 000
maximin: 0			

Figure 2.3. The "SUMUS versus SUNEK" game – Maximin strategy

Meanwhile, given the concept of zero-sum game, the choice of the maximum of the minimums of the columns of the payoff matrix, SUNEK must generate the same strategy which gives the minimum of the maximum in the columns of SUMUS's payoff matrix. Let us consider the following figure:

		<i>SUNEK</i>	
		1 m u	2 m u
<i>SUMUS</i>	1 m u	0	5 000
	2 m u	-5 000	0
		max: 0	max: 5 000
minimax: 0			

Figure 2.4. The "SUMUS versus SUNEK" game – Minimax strategy

It follows that if SUNEK tries to determine the minimum set of maximums – **minimax** – from SUMUS's payoff matrix, they will select the same strategy when trying to find the maximum of the minimums – **maximin** – from the respective payoff matrix. Such strategies, in which the maximum of the minimums of the lines is equal to the minimum of the maximums of the columns is called the **equilibrium point** or **saddle point** of the game, because by choosing these strategies, both companies assure themselves a minimum gain regardless of what the opponent does. Thus, no company will feel motivated to leave its equilibrium strategy unilaterally. Furthermore, no company will have cause to regret their decision as soon as they know their opponent's choice, because they both know that, given the opposing company's choice they would do worse if they took another decision. In other words, the equilibrium solution is stable in the sense that each company may announce its choice before the opponent, assured that the opponent cannot use such information to achieve a higher gain.

## 1.1. ROCK-PAPER-SCISSORS

All of the strategies considered in the previous game were completely deterministic. That is, strategies that establish everything a player should know. Any strategy that is completely deterministic is called ***pure strategy***. An equilibrium where both players use a pure strategy is an ***equilibrium in pure strategies***. However, there are situations in which the equilibrium considers that players use strategies that are not completely deterministic. Any strategy that is not completely deterministic said to be ***mixed strategy***. An equilibrium in which at least one player operates a mixed strategy is said to be ***an equilibrium in mixed strategies***. When players use mixed strategies, they act randomly. The advantage of using mixed strategies is to include uncertainty in the opponent; that is, when player play with mixed strategies they are no longer predictable. Although the goal of a mixed strategy is to keep the opponent in the dark through unpredictability, it does not imply at all a totally random pattern of moves. In a situation where players use mixed strategies, each of them may choose a strategy randomly in each round. Thus, the opponent cannot predict which strategy the player will adopt. Each player's problem will then be to adjust these probabilities optimally.

Mathematically a mixed strategy is a probability distribution over pure strategies. It is through this concept that a game which does not have equilibrium points in pure strategies can be solved, because if any exist they are the game's solution.

To illustrate this, consider the ROCK-PAPER-SCISSORS game. This two-player game, Maximum and Minimum, is played as follows: each player simultaneously makes a gesture representing each of the three objects (rock, paper, scissors). If both players choose the same object, they neither win nor lose; otherwise, victory is achieved according to the following rules: scissors cut paper, paper wraps stone, stone breaks scissors. The payoff is +1 for a win and -1 for a loss. Figure 2.5 represents the normal form of this classic two-player game:

		<i>Minimum</i>		
<i>Maximum</i>		<i>Scissors</i>	<i>Paper</i>	<i>Rock</i>
<i>Scissors</i>		$(0,0)$	$(1,-1)$	$(-1,1)$
<i>Paper</i>		$(-1,1)$	$(0,0)$	$(1,-1)$
<i>Rock</i>		$(1,-1)$	$(-1,1)$	$(0,0)$

Figure 2.5. The “Rock-paper-scissors” game –normal representation

Each player has three pure strategies, Scissors (T) or Paper (P) or Rock (D). Let us take (T,T). Minimum has an incentive to play Rock (D) and, thus, turn a defeat into a victory. The same occurs in each of the nine combinations of pure strategies, where none of the combinations of pure strategies is an equilibrium point. The “scissors-paper-rock” game cannot be solved using pure strategies. This game’s solution necessarily involves mixed strategies.

Let us consider  $p_1$  the probability that Maximum choose scissors,  $p_2$  the probability of choosing paper and  $p_3$ , the probability of choosing rock. Similarly, let us suppose  $q_1$  is the probability of Minimum choosing scissors,  $q_2$ , the probability of choosing paper and  $q_3$ , the probability of choosing rock. Now let us consider Maximum’s payoffs. Assuming Maximum uses pure strategy *Scissors* and Minimum uses a mixed strategy  $q=(q_1,q_2,q_3)$ . As Minimum uses a mixed strategy, Maximum anticipates an expected payoff

$$E(T,q)=q_1 \times 0 + q_2 \times 1 + q_3 \times (-1).$$

When Minimum chooses *Paper* Maximum wins; when chooses Minimum *Rock*, Maximum loses, and when Minimum chooses *Scissors* neither player wins. Let us suppose now that Maximum chooses the pure strategy *Paper*. As Minimum plays a mixed strategy, Maximum anticipates an expected payoff

$$E(P,q)=q_1 \times (-1) + q_2 \times 0 + q_3 \times 1.$$

When Minimum chooses *Scissors*, Maximum loses; when Minimum chooses *Rock*, Maximum wins, and when Minimum chooses paper the players tie. Let us suppose now that Maximum chooses the pure strategy *Rock*. As Minimum uses a mixed strategy, Maximum anticipates an expected payoff

$$E(D,q)=q_1 \times 1 + q_2 \times (-1) + q_3 \times 0.$$

When Minimum chooses *Paper*, Maximum loses; when Minimum chooses *Scissors*, Maximum wins, and when Minimum chooses *Rock*, the players tie

From balancing the various expected payoffs number, we get

$$E(T,q)= E(P,q)= E(D,q) \text{ that is } q_1 \times 0 + q_2 \times 1 + q_3 \times (-1) = q_1 \times (-1) + q_2 \times 0 + q_3 \times 1 = q_1 \times 1 + q_2 \times (-1) + q_3 \times 0$$

Because mixed strategy is a probability we have  $q_1+q_2+q_3=1$ . Solving the system consisting of these equations we obtain  $q_1^*=q_2^*=q_3^*=\frac{1}{3}$ , the value for the Minimum mixed strategy equilibrium. Given the symmetry of the game, the same strategy is an equilibrium for Maximum. In equilibrium each player obtains an expected value of  $\frac{1}{3}$ .

With a simple calculation it is easy to check that if one player maintains equal probabilities for their strategy and the other player changes his set of probabilities, the latter cannot improve his payoff average. We conclude that the strategy of combining equal probability is *an equilibrium point* for the game. In this situation both players can inform their opponent their chosen strategy without incurring any harm.

If a game has a saddle point, the players should not deviate from the strategies that lead to this equilibrium as the pair of strategies such that each player maximizes their respective minimum is the game's solution. When there is no saddle point, being rational players, considering the use of mixed strategies we can use the same criteria to ensure a set of probabilities for each player leading to the same average result, which will be the best payoff each player could get.

This powerful result that von Neumann demonstrated is known as:

**Minimax Theorem:** Any two-player zero-sum game has a mixed strategy for each player, such that the expected gain for both has the same value when players use these strategies. This value is the best gain each player can expect to get, so that such mixed strategies are the optimal strategies for the players.

Thus, for two people in a zero-sum game it is rational for each player to choose the strategy that maximizes the minimum payoff, and the pair of strategies and payoffs such that each player maximizes the minimum. This is the respective game's solution.

Despite attempts by von Neumann and Morgenstern to "extend" this powerful result, that von Neumann demonstrated, to non-constant-sum games with multiple participants, it is only valid for zero-sum games for two players. The greatest difficulty for non-constant sum games with multiple participants led to the fact that of the various solutions presented, none had been accepted mathematically as a solution for non-constant sum games.

## 2. Nash Theorem



The Minimax theorem says what the “rational” solution for two-player zero-sum games is, but it is no solution for games where there is no saddle point. In such cases there will not be a strategy for any of the players that cannot be exploited by an opponent who obtains advance knowledge of what he wishes to do. However, since there is always the possibility of the adversary receiving information about our intentions, how should a rational player proceed under such circumstances? This is the central question supporting all of the mathematical theory of games.

Let us consider another example of price competition based on W. Nutter. Two companies, VILEC and HIPEREL sell “parts” for the price of 1 m u, 2 m u and 3 m u for “parts”. It is assumed that:

- the payoffs are the profits, after all fixed costs are subtracted;
- the company practising lower prices have more customers;
- the company practising lower will prices obtain more profits, with limits, than the company practising the highest price.

The following figure represents the payoff matrix associated with this example:

		VILEC		
		1 m u	2 m u	3 m u
HIPEREL	1 m u	0; 0	50; -10	40; -20
	2 m u	-10; 50	20; 20	90; 10
	3 m u	-20; 40	10; 90	50; 50

Figure 3.1. The “VILEC versus HIPEREL” game –normal representation

From the figure we can see this game is not a zero-sum game. Profits may be 100 m u, 40 m u, 20 m u or 0 m u, depending on the strategy chosen by each company. For this reason the maximin theorem does not apply.

Analysing the payoff matrix from the point of view of the HIPEREL company, they can act as follows: if VILEC chooses a price of 3 m u, the best price HIPERELEC is 2 m u but at this price for HIPEREL, 1 m u will be the best price for VILEC.

Examining the strategy regarding the choice of price of 3 m u for each company, it appears that each can benefit from reducing their price as long as the competitor sticks to their strategy.

Now considering the strategy corresponding to the price of 3 m um for HIPEREL and 2 m u for VILEC, similar reasoning to that above can be made; VILEC can benefit from reducing its price to 1 m u. Following this analysis all strategies are eliminated, except the pair in which both companies set the price

at 1 m u, i.e., the pair of strategies corresponding to the lowest price is such that neither company can improve its payoff through a unilateral change of its strategy.

This example is based on a generalization of the Minimax theorem for the case of non-zero-sum games involving two or more players in direct competition – non-cooperative games. John Nash showed the theorem that generalizes the Minimax theorem:

**Nash Theorem:** Any non-cooperative game of  $n$  players, in which each player has a finite number of pure strategies, has at least one set of equilibrium strategies.

This theorem shows that there can be multiple equilibrium strategies adding great difficulty to what we consider to be rational behaviour. On the other hand, despite being non-cooperative games, the theorem shows that players gain more if they agree to cooperate

### 3. Prisoner's Dilemma

The frontier between pure and applied game theory is vague; some developments in pure theory were motivated by applications. Such is the case of the example that A. W. Tucker presented at a conference addressed to psychologists at Stanford University (1950) with the aim of illustrating the difficulty of analysing non-cooperative games.

In these games it is not possible for players to plan strategies together. They are games that emphasize the rationality required when two individuals are in a position where a decision of one depends on the decision of the other.

Let us consider the following example: two supposed criminals, Joe and Tony are imprisoned. The problem for the police is, assuming that both are involved and in the absence of evidence, the need for a confession. The prisoners are in individual and distant cells with no communication between them. Each receives an explanation of the rules of the case:

- If neither of them chooses to confess both will be charged with a misdemeanour that involves a symbolic penalty of only one month in prison.
- If both confess to taking part in the crime, then they will both be sentenced to six months in prison.
- Finally, if one confesses and the other does not, then whoever confesses will be released immediately, and the other will be sentenced to the maximum sentence under the law: nine months in prison (six months for the crime plus three more for obstructing justice).

The strategies in this case are: to confess or not to confess. The payoffs are the sentences. We can express this example using the following payoff matrix:

		<i>Tony</i>	
		<i>Does not confess</i>	<i>Confesses</i>
<i>Joe</i>	<i>Does not confess</i>	<i>(1,1)</i>	<i>(9,0)</i>
	<i>Confesses</i>	<i>(0,9)</i>	<i>(6,6)</i>

*Figure 3.1 - Prisoner's dilemma –normal representation*

The matrix reads as follows: each prisoner chooses one of two strategies. Joe chooses a line while Tony chooses a column. Both numbers in each cell express each prisoner's sentence and correspond to the pair of strategies chosen by them.

The number on the left corresponds to the payoff of the prisoner who chooses lines – Joe, while the number on the right corresponds to the payoff of the prisoner who chooses columns – Tony. Thus, reading the first column in descending order, if neither confesses, each is sentenced to a sentence of 1 month, but if Joe confesses and Tony does not, Joe goes free, while Tony is sentenced to 9 months.

Which will be the "rational" strategies so that each of the criminals minimizes the time he will spend in prison? Two things can happen: Joe confesses or does not confess. Now, if Joe confesses, we have:

	<i>Tony does not confess</i>	<i>Tony is sentenced to 9 months in prison</i>
<i>Joe confesses</i>	<i>Tony confesses</i>	<i>Tony is sentenced to 6 months in prison</i>

*Figure 4.2- Prisoner's dilemma – Joe confesses*

If Joes does not confess, we get:

	Tony does not confess	Tony is sentenced to 1 month in prison
Joe does not confess	Tony confesses	Tony goes free

Figure 4.3- Prisoner's dilemma – Joe does not confess

Observing the previous two tables we find that, in both cases, it is better for Tony to confess. Reversing Joe and Tony's roles, it will also be better for Joe to confess. The result of the game will be one in which both prisoners confess to the crime, both prisoners' rationality makes them choose the strategy of confessing the crime.

This is due to the fact that both criminals are facing **dominant equilibrium strategy**; regardless of the combinations of strategies that each of criminals does, the best choice is always "to confess" and thus called a dominant strategy. Since both players "play" the dominant strategy, they "fall" into the dominant strategy equilibrium.

The Prisoners' Dilemma game is a two player game that is based on the conflict between individual and collective rationality. According to individual rationality, the prisoner gets a higher payoff by denouncing the other. If both prisoners sacrifice themselves through silence, they get a better payoff. So, to ensure the best payoff, prisoners are base their decisions on common interests. In this game, players do not communicate with each other and play only once. Since these are changed assumptions, it is most likely that the outcome of the game will also change.

#### 4. Conclusion

Even when Tucker thought up the Prisoners' Dilemma, game theory was already a recognized science. We can say that those responsible for this recognition were John von Neumann and Oskar Morgenstern with the publication of the book "Theory of Games and Economic Behaviour" (1944). John Nash (Nobel prize 1994), was also a pioneer in this science by having formalized clearly the types of games and their possibility for equilibrium. Later Nash's results were successively extended to more complex cases, with crucial steps in this process taken by Reinhard Selten and John Harsanyi (Nobel prize 1994).

Today game theory is well advanced, enabling vast and interesting results to be obtained in classifying, formalizing and solving day-to-day conflicts in all areas and in all situations involving strategic interaction.

However, because the assumptions impose constraints which guide the actions of the players involved, and do not observe their personalities, there is still much to do. With no intention of discrediting the techniques and analyses studied, of course, its limitations are mentioned in order to allow the reader to gain a clear awareness of the limitations of the analytical methods studied, because without that “we may become their slaves rather than their masters”.

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