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The Riemann Hypothesis

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Abstract. Let's define $\delta(n) = (\sum_{q \leq n} \frac{1}{q} - \log \log n - B)$, where $B \approx 0.2614972128$ is the Meissel-Mertens constant. The Robin theorem states that $\delta(n)$ changes sign infinitely often. We prove if the inequality $\delta(p) \leq 0$ holds for a prime p big enough, then the Riemann Hypothesis should be false. However, we could restate the Mertens second theorem as $\lim_{n \to \infty} \delta(p_n) = 0$ where p_n is the n^{th} prime number. In this way, this work could mean a new step forward in the direction for finally solving the Riemann Hypothesis.

1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log \log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime q divides to N_n. The importance of this property is:

Theorem 1.1 *[5], [6].* $\mathsf{Nicolas}(p_n)$ *holds for all prime* $p_n > 2$ *if and only if the Riemann Hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We use the following property of the Chebyshev function:

Theorem 1.2 *[2]. For a prime* p *big enough:*

$$\theta(p) = (1 + o(1)) \times p.$$

Let's define $S(x) = \theta(x) - x$. Nicolas also proves that

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Theorem 1.3 *[6].* $\forall x \geq 121$*:*

$$\log\log\theta(x) \geq \log\log x + \frac{S(x)}{x \times \log x} - \frac{S(x)^2}{x^2 \times \log x}.$$

The famous Mertens paper provides the statement:

Theorem 1.4 *[4].*

$$\log\left(\prod_{q \leq n} \frac{q}{q-1}\right) = \sum_{q \leq n} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > n} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > n} \frac{1}{q^3} - \cdots$$

where $B \approx 0.2614972128$ *is the Meissel-Mertens constant.*

Let's define:

$$\delta(n) = \left(\sum_{q \leq n} \frac{1}{q} - \log\log n - B\right),$$

Robin theorem states the following result:

Theorem 1.5 *[7].* $\delta(n)$ *changes sign infinitely often.*

In addition, the Mertens second theorem states that:

Theorem 1.6 *[4].*

$$\lim_{n \to \infty} \delta(n) = 0.$$

Putting all together yields the proof that when the inequality $\delta(p) \leq 0$ holds for a prime p big enough, then the Riemann Hypothesis should be false.

2 Central Lemma

Lemma 2.1 *For a prime* p *big enough:*

$$0 < \frac{S(p)}{p} < 1.$$

Proof By the theorem 1.2, for a prime p big enough:

$$\begin{aligned}
\frac{S(p)}{p} &= \frac{\theta(p) - p}{p} \\
&= \frac{(1 + o(1)) \times p - p}{p} \\
&= \frac{p \times ((1 + o(1)) - 1)}{p} \\
&= (1 + o(1) - 1) \\
&= o(1).
\end{aligned}$$

Since $0 < o(1) < 1$, then the proof is finished. ∎

3 Main Theorem

Theorem 3.1 *If the inequality $\delta(p) \leq 0$ holds for a prime p big enough, then the Riemann Hypothesis should be false.*

Proof For a prime p big enough, suppose that simultaneously $\mathsf{Nicolas}(p)$ and $\delta(p) \leq 0$ hold. If $\mathsf{Nicolas}(p)$ holds, then

$$\prod_{q \leq p} \frac{q}{q-1} > e^{\gamma} \times \log \theta(p).$$

We apply the logarithm to the both sides of the inequality:

$$\log \left(\prod_{q \leq p} \frac{q}{q-1} \right) > \gamma + \log \log \theta(p).$$

We use that theorem 1.4:

$$\log \left(\prod_{q \leq p} \frac{q}{q-1} \right) = \sum_{q \leq p} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots .$$

Besides, we use that theorem 1.3:

$$\log \log \theta(p) \geq \log \log p + \frac{S(p)}{p \times \log p} - \frac{S(p)^2}{p^2 \times \log p}.$$

Putting all together yields the result:

$$\begin{aligned} &\sum_{q \leq p} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots \\ &> \gamma + \log \log \theta(p) \\ &\geq \gamma + \log \log p + \frac{S(p)}{p \times \log p} - \frac{S(p)^2}{p^2 \times \log p}. \end{aligned}$$

Let distribute it and remove γ from the both sides:

$$\sum_{q \leq p} \frac{1}{q} - \log \log p - B - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots >$$

$$\frac{1}{\log p} \times \left(\frac{S(p)}{p} - \frac{S(p)^2}{p^2} \right).$$

We know that $\delta(p) = \sum_{q \leq p} \frac{1}{q} - \log \log p - B$. Moreover, we know that $\left(\frac{S(p)}{p} - \frac{S(p)^2}{p^2} \right) > 0$. Indeed, according to the lemma 2.1, we have that $0 < \frac{S(p)}{p} < 1$. Consequently, we obtain that $\frac{S(p)}{p} > \frac{S(p)^2}{p^2}$ since for every real number $0 < x < 1$, the inequality $x > x^2$ is always satisfied. To sum up, we would have that

$$\delta(p) - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots > 0$$

because of

$$\frac{1}{\log p} \times \left(\frac{S(p)}{p} - \frac{S(p)^2}{p^2} \right) > 0.$$

However, the inequality

$$\delta(p) - \frac{1}{2} \times \sum_{q>p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q>p} \frac{1}{q^3} - \cdots > 0$$

never holds when $\delta(p) \leq 0$. By contraposition, $\mathsf{Nicolas}(p)$ does not hold when $\delta(p) \leq 0$ for a prime p big enough, In conclusion, if $\mathsf{Nicolas}(p)$ does not hold for a prime p big enough, then the Riemann Hypothesis should be false due to the theorem 1.1. ■

4 Discussion

The Riemann Hypothesis has been qualified as the Holy Grail of Mathematics [3]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [3]. In the theorem 3.1, we show that if the inequality $\delta(p) \leq 0$ holds for a prime p big enough, then the Riemann Hypothesis should be false. Nevertheless, the well-known theorem 1.6 could be restated as

$$\lim_{n \to \infty} \delta(p_n) = 0$$

because of there are infinitely many prime numbers p_n. Indeed, we think this work could help the scientific community in the global efforts for trying to solve this outstanding and difficult problem.

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