An Augmented Lagrangian Formulation of the Planar Elastica in Constrained Environments

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Introduction

Slender and highly flexible structures can be found in multiple engineering applications. Such is the case of flexible endoscopes, which is of particular interest to us. Due to the geometrically highly non-linear behaviour of these structures, it is crucial to model these systems accordingly. Moreover, these devices work in narrow and confined spaces and it is important to properly handle contact in simulations. In this work, we focus on the contact problem of a 2-dimensional beam model, whose deformation is confined by narrow and tortuous surroundings. To describe this problem, we propose a discrete augmented Lagrangian formulation of Euler's *elastica* with unilateral contact potentials as in [1]. Then, a variational integrator is derived by applying a discrete version of Hamilton's principle of stationary action [5].

Planar Euler's elastica

We consider a planar inextensible beam model, i.e., Euler's elastica, in which the cross-section is assumed to be constant along the arc-length s and perpendicular to the centreline $q(s)$. Then, the deformation of the centreline is a pure bending problem described by the following constrained second-order Lagrangian $\widehat{L}: T^{(2)}\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$

$$\widehat{L}\left(q,q',q'',\Lambda\right) = L\left(q,q',q''\right) + \Lambda\Phi(q,q') = \frac{1}{2}EI\left\|q''\right\|^2 + \Lambda\Phi(q,q'), \tag{1}$$

where, $q'(s)$ and $q''(s)$ denote the first and second spatial derivatives of q, respectively, EI is the bending stiffness, $\Lambda(s)$ is a Lagrange multiplier and $\Phi(q,q')$ is the constraint function in Eq. 2, which guarantees both the inextensibility and the arc-length parametrisation of the beam [4, 2].

$$\Phi(q,q') = \|q'\|^2 - 1 \tag{2}$$

Taking variations subject to the boundary conditions $(q(0),q'(0)) = (q_0,q'_0)$ and $(q(l),q'(l)) = (q_N,q'_N)$, where $l \in \mathbb{R}$, $l > 0$ is the length of the beam, yields the static equilibrium equations of the problem in the form of Euler-Lagrange equations. For more details on the beam model and the spatial discretisation of the problem see [3].

Unilateral contact problem

A further augmented Lagrangian $\widetilde{L}$, presented in [1] as in Eq. 3, describes the problem of the elastica in contact with a rigid wall.

$$\widetilde{L} = \widehat{L} - kg\Lambda_c + \frac{1}{2}pg^2 - \frac{1}{2p}\left(dist\left(k\Lambda_c - pg, \mathbb{R}^+\right)\right)^2 \tag{3}$$

Here, k is a scaling factor, $g(s)$ is the gap function w.r.t. the wall, $\Lambda_c(s)$ is a Lagrange multiplier related to the contact problem, and p is a positive penalty coefficient. This model describes the unilateral contact problem, which implies impenetrability ($g(s) \geq 0$), complementarity ($\Lambda_c(s)\, q(s) = 0$) and no traction forces between the bodies ($\Lambda_c(s) \leq 0$). In this work, the gap functions $g(s)$ used to model the rigid walls are

$$\begin{aligned} g(q) &= r_1 - q && \text{for a straigth wall}\,, \\ g(q) &= \|\sqrt{r_2^2 - (x_{circle} - x_C)^2} + y_C\| - \|q\| && \text{for a circular wall}\,, \end{aligned} \tag{4}$$

where, r_1 defines the distance of a straight wall from the horizontal axis, r_2 is the radius of a semicircle, x_{circle} represents the x-coordinates of a curved wall, (x_C, y_C) are the coordinates of the centre of a semicircle.

Conclusions and remarks

Fig. 1 shows the deformed configuration of the elastica for given boundary conditions in blue in contact with a straight narrow tube on the left, and a curved rigid wall on the right. More complex and narrow geometrical shapes are of particular interest when using the presented contact Lagrangian formulation.

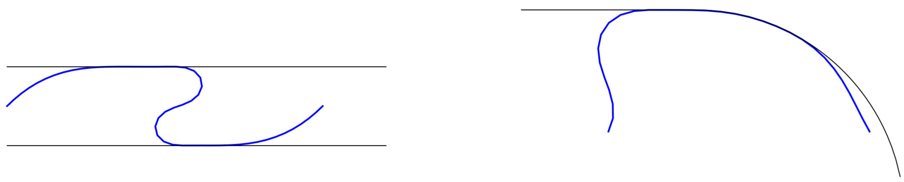

Figure 1: Static equilibria of a planar elastica of fixed length and fixed initial and final positions and slopes in contact with a straight narrow tube on the left and a circular elbow on the right.

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