Generalizing the concept of quantum triads

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Abstract

The concept of *quantum triad* has been introduced by D. Kruml [6], where for a given pair of quantale[1] modules L, R over a common quantale Q, endowed with a bimorphism (a 'bilinear map') to Q, a construction equipping L and R with additional module structure and another bimorphism, both compatible with the existing bimorphism and action of the quantale, was presented. As the original concept was only defined in a specific setting of categories of quantale modules, we extend it to a more universal one, which can be applied to other common algebraic structures.

In what follows, we assume that all the categories are concrete (via the forgetful functor $|-|$ into **Set**). Let $\mathbf{V} = (\mathbf{V}, \otimes_{\mathcal{V}}, I_{\mathbf{V}})$ be a closed symmetric monoidal category and $\mathbf{C}$ be a subcategory of $\mathbf{V}$, enriched in $\mathbf{V}$. Suppose M is a monoid in $\mathbf{V}$ (viewed as a $\mathbf{V}$-category with a single object $\star_M$). We call an object A of $\mathbf{C}$ a left (right) module over M if there is a $\mathbf{V}$-enriched functor from M (M^{op}) to $\mathbf{C}$ that maps $\star_M$ to A. Next, A is a M, N-bimodule if there is a $\mathbf{V}$-enriched bifunctor from $M \times N^{op}$ to $\mathbf{C}$ mapping $(\star_M, \star_N)$ to A. A map (in **Set**) $f: |A| \times |B| \to |C|$ is a $\mathbf{C}$-bimorphism[1] if every map $f(a, -): |B| \to |C|$ and $f(-, b): |A| \to |C|$ arises from a $\mathbf{C}$-morphism $f_A: B \to C$ or $f_B: A \to C$, respectively, as $|f_A|$ or $|f_B|$.

Definition. *Let $\mathbf{V}, \mathbf{C}$ be as above, L, R be a left and a right module over a $\mathbf{V}$-monoid T. Further, let τ be a $\mathbf{C}$-bimorphism from $L \times R$ to T. Then the tuple (L, R, τ, T) is called a* triad. *If there exists a monoid S in $\mathbf{V}$ together with a $\mathbf{V}$-bimorphism $\sigma: R \times L \to S$ which makes L a T, S-bimodule and R a S, T-bimodule, and is compatible with τ (this means, for instance 'RLR': for any $l \in L$ and $r, r' \in R$, $\sigma(r, l) \cdot r' = r \cdot \tau(l, r')$, and a few similar conditions), we call (S, σ) a* solution *of the triad.*

Existence of solutions together with additional properties can be proved when certain assumptions on the category $\mathbf{C}$ are satisfied:

Proposition.   *1. Let $\mathbf{C}$ have tensor products over T, i.e., it has coequalizers of morphisms $R \otimes T \otimes L \rightrightarrows R \otimes L$ obtained from T acting on R and L, respectively. Then the universal property of the tensor product provides a solution (S_0, σ_0) given by $R \otimes_T L$ and $\sigma_0: (r, l) \mapsto r \otimes_T l$, which is initial – for any solution (S, σ) there is a morphism $s_0: S_0 \to S$ such that $\sigma = s_0 \circ \sigma_0$. In this case, the solution belongs to $\mathbf{C}$. Multiplication in S_0 and action of S_0 on R are as follows:*

- *$(r_1 \otimes_T l_1) \cdot (r_2 \otimes_T l_2) = r_1 \cdot \tau(l_1, r_2) \otimes_T l_2 = r_1 \otimes_T \tau(l_1 r_2) \cdot l_2$,*
- *$(r \otimes_T l) \cdot r' = r \cdot \tau(l, r')$ and $l' \cdot (r \otimes_T l) = \tau(l, r') \cdot l$.*

2. $S_1 = \{(\alpha, \beta) \in \mathbf{C}(L, L) \times \mathbf{C}(R, R) \mid \tau(\alpha(l), r) = \tau(l, \beta(r))$ for any $l \in L, r \in R\}$ with $\sigma_1: (l, r) \mapsto ((-l)r, l(r-))$ is another solution. It is terminal, since any monoid acting on L (R) and satisfying the compatibility conditions can be represented in S_1. Multiplication and action of S_1 are following:

- $(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$,
- $(\alpha, \beta)r = \beta(r)$ *and* $l(\alpha, \beta) = \alpha(l)$.

It has been shown in [6] that S_0 and S_1 form a so-called couple of quantales (see also [4]) and that S is a solution iff $S_0 \to S_1$ factorizes through S.

An example application in the original paper was reconstructing a quantale from given sets of its right-, left- and two-sided elements, and the base category was **SupLat**, complete join-semilattices with join-preserving maps. Here we extend this construction to other categories.

Example 1. Let V be a linear space over a field K, equipped with an inner product $\langle -, - \rangle \colon V \times V \to K$. The solutions of the triad $(V, V, \langle -, - \rangle, K)$ are then provided by linear operators on V, namely S_0 consisting of elements $u_1 \otimes_K v_1 + \cdots + u_n \otimes_K v_n$, each acting on V as making a linear combination of $u_1, \ldots, u_n$ with coefficients $\langle v_i, w \rangle$, and $S_1 = \{(\alpha, \beta) \in \mathrm{End}(V) \times \mathrm{End}(V) \mid \langle \alpha(u), v \rangle = \langle u, \beta(v) \rangle\}$, the ring formed by pairs of adjoint linear operators on V (with reversed order of composition on the first component).

In many situations, the categories **C** and **V** coincide, like in the original case of quantales (with **SupLat**, the category of complete join-semilattices), rings (with **Ab**, the category of abelian groups), or ordinary monoids (with **Set**), and the whole category of solutions is contained in **C**.

Using more recent results on existence of tensor products for some algebras of quantum logic, other categories allowing formation of triads can be considered. For instance, we take the category effect algebras[2] with effect algebra morphisms, **EffAlg**. Here the distinction of **V** and **C** is required as **EffAlg** has tensor products (when trivial effect algebras with $0 = 1$ are allowed) and is monoidal with the 2-element algebra $\{0, 1\}$ being the tensor unit, however, it is not closed [5]. The enriching category **V** is now that of generalized effect algebras (**GenEffAlg**).

Example 2. A state ('probability measure') on an effect algebra E is an effect-algebra morphism (preserving $\oplus$ if it is defined, $^\perp$, 0 and 1) $s \colon E \to [0, 1]$. A convex effect algebra is an effect algebra with action of the real interval $[0, 1]$ (viewed as an effect algebra, with $a \oplus b$ defined if $a + b \leq 1$), which becomes a monoid in **EffAlg** under standard multiplication of reals. Having two convex effect algebras with states (E, s_E) and (F, s_F), one obtains an effect algebra with a state $(E \otimes F, s_E \cdot s_F)$ [2]. This forms a triad $E \times F \to [0, 1]$. Like in the previous example, its special solutions are $S_0 = F \otimes_{[0,1]} E$, and S_1 – pairs of endomorphisms acting on the components of the tensor product, mutually adjoint wrt. the state on $E \otimes F$.

References

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[2] Briefly, a partial commutative monoid (pcmon) is an algebra $(M, \oplus, 0)$ where $0 \in M$ and $\oplus$ is a partially defined commutative and associative operation with 0 as a neutral element. A conical and cancellative pcmon is called a *generalized effect algebra*. A pcmon E is called an *effect algebra* if it is equipped with a unary operation $^\perp$ and an element 1 such that $x \oplus y = 1$ iff $y = x^\perp$, and having $x \oplus 1$ defined implies $x = 0$. A symmetric binary relation (orthogonality) can be defined on E by $a \perp b$ iff $a \oplus b$ is defined. A frequent instance of effect algebras is the one of bounded positive operators on a Hilbert space. For reference on effect algebras and other quantum structures, we suggest [3].

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