Note on Deduction Theorems in Contraction-Free Logics

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Abstract

In this short paper we present a finer analysis of the variants of Local Deduction Theorem in contraction-free logics. We define some natural generalisations called Implicational Deduction Theorems and study their basic properties. The hierarchy of classes of logics defined by these theorems is presented.

1 Introduction

One of the most important theorems of classical propositional logic is the Deduction Theorem, independently discovered by Herbrand [3] and Tarski [5], which connects provability and implication. In its most popular form it says

$$\Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash \varphi \to \psi.$$

It enables us to find some proofs much easier. However, this theorem does not hold in all logics. For example in logics without contraction, we usually have only a (form of) Local Deduction Theorem, which says that there exists some natural k such that

$$\Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash \underbrace{\varphi \to (\ldots(\varphi}_{k\text{-times}} \to \psi)\ldots).$$

The problem is that generally we do not have any (reasonable) upper bound on k.

We can try to give some estimates of k. The immediate idea is to count how many times the assumption φ is used in the proof of ψ. This idea is captured in the forthcoming paper [2] where the situation is shown not to be so easy. We define some hierarchy of logics with Implicational Deduction Theorems and investigate relations between its members. It is shown that this hierarchy collapses on some level and its first four members are not the same.

For further details, proofs and references we refer the reader to the forthcoming paper [2].

2 Preliminaries

We use some standard terminology from the theory of logical calculi (see e.g. [6])—a *propositional language* $\mathcal{L}$, the set of $\mathcal{L}$*-formulae* $Fle_{\mathcal{L}}$ over some fixed countably infinite set of propositional variables and $\mathcal{L}$*-substitutions*. In this paper we assume that $\mathcal{L}$ always contains a binary connective called implication $\to$ and we use the following convention:

$$\varphi \to^0 \psi = \psi \qquad \text{and} \qquad \varphi \to^{i+1} \psi = \varphi \to (\varphi \to^i \psi).$$

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An $\mathcal{L}$-*theory* Γ is a set of $\mathcal{L}$-formulae. A *(finite)* $\mathcal{L}$-*consecution* $\Gamma \rhd \varphi$ is a pair consisting of a (finite) theory Γ and a formula φ.

A *logic* $\mathbf{L}$ in the language $\mathcal{L}$ is a structural consequence relation (in the sense of Tarski) on $Fle_{\mathcal{L}}$. That is, $\mathbf{L}$ is a set of relations between theories and formulae (writing $\Gamma \vdash_{\mathbf{L}} \varphi$, and $\Gamma \vdash_{\mathbf{L}} \Gamma'$ as an abbreviation for $\Gamma \vdash_{\mathbf{L}} \varphi$ for each $\varphi \in \Gamma'$) satisfying the following conditions:

(i) If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
(ii) If $\Gamma \vdash_{\mathbf{L}} \Gamma'$ and $\Gamma' \vdash_{\mathbf{L}} \varphi$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
(iii) If $\Gamma \vdash_{\mathbf{L}} \varphi$, then there is a finite set $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash_{\mathbf{L}} \varphi$.
(iv) If $\Gamma \vdash_{\mathbf{L}} \varphi$, then $\sigma(\Gamma) \vdash_{\mathbf{L}} \sigma(\varphi)$ for any $\mathcal{L}$-substitution σ.

The previous conditions are called reflexivity, cut, finitarity and structurality.

A logic $\mathbf{L}_2$ in a language $\mathcal{L}_2 \supseteq \mathcal{L}_1$ is an *expansion* of $\mathbf{L}_1$ in $\mathcal{L}_1$ if for each $\mathcal{L}_1$-theory Γ and $\mathcal{L}_1$-formula φ: $\Gamma \vdash_{\mathbf{L}_1} \varphi$ implies $\Gamma \vdash_{\mathbf{L}_2} \varphi$.

Definition 1. *An* axiomatic system AX *is a set of finitary consecutions closed under substitutions. The members of* AX *with non-empty theories are called* deduction rules*, these with empty theories are called* axioms*. We say that* AX *is* MP-based *if modus ponens is its only deduction rule.*

Note that we only have finitary rules, and axioms as well as rules are presented by schemata.

Definition 2. *Let* AX *be an axiomatic system. An* AX-proof *of formula* φ *in theory* Γ *is a finite tree labelled by formulae satisfying*

(i) the root is labelled by φ*,*
(ii) leaves are labelled either by axioms or by elements of Γ*,*
(iii) if a node is labelled by ψ *and its preceding nodes are labelled by* $\psi_1, \dots, \psi_n$ *then* $\{\psi_1, \dots, \psi_n\} \rhd \varphi \in \mathsf{AX}$.

If such a proof exists we write $\Gamma \vdash_{\mathsf{AX}} \varphi$.

We say that AX is an *axiomatic system* for (a presentation of) a logic $\mathbf{L}$ iff $\mathbf{L} = \vdash_{\mathsf{AX}}$. A logic $\mathbf{L}$ is *MP-based* if it has some MP-based presentation.

The well-known logic BCI by C. A. Meredith (cf. [4]) is a generalisation of logic BCK.

Definition 3. *The logic BCI has the following axioms:*

(B) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$,
(I) $\varphi \to \varphi$.

The only deduction rule is modus ponens.

3 Implicational Deduction Theorems

In this section we define Implicational Deduction Theorems. First one will be a form of the Local Deduction Theorem mentioned in the introduction.

Definition 4. *A logic* **L** *has* Simple Implicational Deduction Theorem *(IDT$_0$) if for each theory Γ and formulae φ, ψ:*

$$\Gamma, \varphi \vdash_{\mathbf{L}} \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_{\mathbf{L}} \varphi \to^n \psi.$$

We immediately obtain the following important property of logics with IDT$_0$, which is a consequence of our assumptions concerning finitarity.

Lemma 1. *A logic* **L** *with IDT$_0$ is MP-based.*

Now we present a finer analysis of Local Deduction Theorems arising from the idea of counting number of occurrences of φ in the leaves of some proof of ψ in Γ and φ.

Definition 5. *Let $n > 0$. A logic* **L** *has n-*Implicational Deduction Theorem *(IDT$_n$) if*

(i) **L** *has an MP-based presentation* AX*,*

(ii) for each theory Γ, formula ψ, mutually different formulae φ_i, $1 \leq i \leq n$, and for each AX-proof $\mathcal{P}$ of ψ in $\Gamma \cup \{\varphi_i \mid 1 \leq i \leq n\}$:

$$\Gamma \vdash \varphi_1 \to^{j_1} (\varphi_2 \to^{j_2} \ldots (\varphi_n \to^{j_n} \psi) \ldots),$$

where j_i is the number of occurrences of φ_i in the leaves of $\mathcal{P}$.

It may seem that e.g. IDT$_2$ can be obtained just by double application of IDT$_1$, but it is not true.

Example 1. *Let us assume that*

$$\varphi, \psi, \varphi \to (\psi \to \chi) \vdash \chi. \tag{1}$$

Clearly there is a proof of (1) *using all premises exactly once. IDT$_2$ gives*

$$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi) \tag{2}$$

and IDT$_1$ gives

$$\psi, \varphi \to (\psi \to \chi) \vdash \varphi \to \chi, \tag{3}$$

but now we cannot use IDT$_1$ once again to obtain (2)*. We only know that $\varphi \to \chi$ is provable from ψ and $\varphi \to (\psi \to \chi)$, but we do not know how many times ψ has to be used.*

From now on, we shall denote the class of all logics satisfying IDT$_n$ also by IDT$_n$. Clearly, if a logic **L** has IDT$_n$ then **L** has IDT$_m$ for any $m \leq n$, i.e., the classes of logics IDT$_n$ form a decreasing chain. In [2] we prove the following interesting theorem about the characterisation of IDT$_3$ and strictness of inclusions in the chain.

Theorem 1.

(i) A logic **L** *has IDT$_3$ if and only if* **L** *is an MP-based expansion of BCI.*

(ii) If a logic **L** *has IDT$_3$ then* **L** *has IDT$_m$ for any $m \geq 3$.*

(iii) IDT$_0 \neq$ IDT$_1$, IDT$_1 \neq$ IDT$_2$, and IDT$_2 \neq$ IDT$_3$.

Let us remark that the counterexample which proves IDT$_2 \neq$ IDT$_3$ was found with the help of a computer and the detailed proof can be found in [1].

Perhaps surprisingly, the previous theorem shows that the hierarchy of logics with Implicational Deduction Theorems has the following form

$$\mathrm{IDT}_0 \supsetneq \mathrm{IDT}_1 \supsetneq \mathrm{IDT}_2 \supsetneq \mathrm{IDT}_3 = \mathrm{IDT}_4 = \cdots = \{\text{MP-based expansions of BCI}\}.$$

4 Summary

We defined a natural sequence of Implicational Deduction Theorems, which generalize the usual Local Deduction Theorem of contraction-free logics, and presented a decreasing hierarchy of logics satisfying them. We showed that only the first four members of this sequence are mutually different and IDT_j for any $j \geq 3$ coincide with the class of MP-based expansions of the logic BCI.

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