An analogue of Bull's theorem for Hybrid Logic (Extended Abstract)

Willem Conradie[1] and Claudette Robinson[1]

Department of Mathematics University of Johannesburg, South Africa
wconradie@uj.ac.za, claudette.robinson574@gmail.com

Hybrid logic extends modal logic with a special sort of variables, called *nominals*, which are evaluated to singletons in Kripke models by valuations, thus acting as names for states in models, see e.g. [1]. Various syntactic mechanisms for exploiting and enhancing the expressive power gained through the addition of nominals can be included, most characteristically the *satisfaction operator*, $@_{\mathbf{i}}\varphi$, allowing one to express that φ holds at the world named by a nominal $\mathbf{i}$.

In [3], R.A. Bull famously proved that each normal extension of **S4.3** has the finite model property. In the current paper we prove a hybrid analogue of Bull's result. Like the proof of Bull's original result, ours is algebraic, and thus our secondary aim with this work is to illustrate the usefulness of algebraic methods within hybrid logic research, a field where such methods have been largely ignored (with the exception of T. Litak's algebraization [5] of a very expressive hybrid logic with binders, using algebras closely akin to cylindric algebras).

1 Syntax and algebraic semantics

We fix two countably infinite, disjoint sets PROP and NOM of propositional variables and *nominals*, respectively. The syntax of the language $\mathcal{H}(@)$ is given as follows:

$$\varphi ::= \bot \mid p \mid \mathbf{j} \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid @_{\mathbf{j}}\varphi,$$

where $p \in \mathsf{PROP}$ and $\mathbf{j} \in \mathsf{NOM}$.

Definition 1.1 (Normal Hybrid extensions of **S4.3**)**.** For any set of $\mathcal{H}(@)$-formulas Σ, the logic $\mathbf{LP}\Sigma$ is the smallest set of formulas containing Σ, the axioms in Table 1 and closed under the inference rules in Table 1, except for $(Name_{@})$ and (BG). $\mathbf{LP}^{+}\Sigma$ is defined similiarly, closing in addition under $(Name_{@})$ and (BG).

Algebraically $\mathbf{LP}\Sigma$ is characterized by classes of CSADAs:

Definition 1.2. A *closure satisfaction algebra with a designated set of atoms* (CSADA) is a pair $\mathfrak{A} = (\mathbf{A}, X)$, where X is a non-empty subset of atoms of $\mathbf{A}$ and $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond, @)$ such that $(A, \wedge, \vee, \neg, \bot, \top)$ is a Boolean algebra, $@$ is a binary operator whose first coordinate ranges over NOM and the second coordinate over all elements of the algebra, and for all $x, y \in A$ and all $u, v, w \in X$ the following holds:

$$\Diamond(x \vee y) = \Diamond x \vee \Diamond y \qquad \Diamond\bot = \bot$$
$$x \leq \Diamond x \qquad \Diamond\Diamond x \leq \Diamond x$$
$$\Diamond x \wedge \Diamond y \leq \Diamond(x \wedge \Diamond y) \vee \Diamond(x \wedge y) \vee \Diamond(y \wedge \Diamond x) \qquad @_u(\neg x \vee y) \leq \neg @_u x \vee @_u y$$
$$\neg @_v x = @_v \neg x \qquad @_u @_v x \leq @_v x$$
$$@_v v = \top \qquad v \wedge x \leq @_v x$$
$$\Diamond @_v x \leq @_v x \qquad @_u \Diamond v \wedge @_w \Diamond v \leq @_u \Diamond w \vee @_w \Diamond u$$

Axioms:

$(Taut)$	$\vdash \varphi$ for all propositional tautologies φ.
(K)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
(4)	$\vdash \Diamond\Diamond p \to \Diamond p$
(T)	$\vdash p \to \Diamond p$
$(.3)$	$\vdash \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(q \wedge \Diamond p)$
$(K_@)$	$\vdash @_\mathbf{j}(p \to q) \to (@_\mathbf{j} p \to @_\mathbf{j} q)$
$(Selfdual)$	$\vdash \neg @_\mathbf{j} p \leftrightarrow @_\mathbf{j}\neg p$
(Ref)	$\vdash @_\mathbf{j}\mathbf{j}$
$(Intro)$	$\vdash \mathbf{j} \wedge p \to @_\mathbf{j} p$
$(Back)$	$\vdash \Diamond @_\mathbf{j} p \to @_\mathbf{j} p$
$(Agree)$	$\vdash @_\mathbf{i} @_\mathbf{j} p \to @_\mathbf{j} p$
$(.3^{-1})$	$\vdash @_\mathbf{i}\Diamond\mathbf{j} \wedge @_\mathbf{k}\Diamond\mathbf{j} \to @_\mathbf{i}\Diamond\mathbf{k} \vee @_\mathbf{k}\Diamond\mathbf{i}$

Rules of inference:

$(Modus\ ponens)$	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$
$(Sorted\ substitution)$	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(Nec)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
$(Nec_@)$	If $\vdash \varphi$, then $\vdash @_\mathbf{j}\varphi$.
$(Name_@)$	If $\vdash @_\mathbf{j}\varphi$, then $\vdash \varphi$ for $\mathbf{j}$ not occurring in φ.
(BG)	If $\vdash @_\mathbf{i}\Diamond\mathbf{j} \to @_\mathbf{j}\varphi$, then $\vdash @_\mathbf{i}\Box\varphi$ for $\mathbf{j} \neq \mathbf{i}$ and $\mathbf{j}$ not occurring in φ.

Table 1: Axioms and inference rules of **LP** and $\mathbf{LP}^+$

$\mathcal{H}(@)$-terms are interpreted in CSADAs $(\mathbf{A}, X)$ in the usual way but subject to the constraint that nominals range over X, while the propositional variables range over all elements of the algebra, as usual.

Theorem 1.3. *Every logic* $\mathbf{LP}\Sigma$ *is sound and complete with respect to the class of all CSADAs validating* Σ*.*

Definition 1.4. A *permeated modal algebra (PCSA)* is a CSADA $\mathfrak{A} = (\mathbf{A}, X)$ such that

1. for each $\bot \neq b \in A$ there is an atom $a \in X$ such that $a \leq b$, and
2. for all $a \in X$ and $b \in A$, if $a \leq \Diamond b$ then there exists an $a' \in X$ such that $a' \leq b$ and $a \leq \Diamond a'$.

Theorem 1.5. *Every logic* $\mathbf{LP}^+\Sigma$ *is sound and complete with respect to the class of all PCSAs validating* Σ*.*

2 Finite model property

In this section, we give an outline of the proof of our main result:

Theorem 2.1. *Every normal hybrid logic* $\mathbf{LH}\Sigma$ *is characterized by the class of all finite CSADAs validating* Σ.

The goal is to find a CSADA refuting a given non-theorem of $\mathbf{LP}\Sigma$. Suppose $\nvdash_{\mathbf{LP}\Sigma} \varphi$, then $\nvdash_{\mathbf{LP}^{+}\Sigma} \varphi$, since $\mathbf{LP}\Sigma$ and $\mathbf{LP}^{+}\Sigma$ have the same theorems. By theorem 1.5, there is a PCSA $\mathfrak{A} = (\mathbf{A}, X)$ and an assignment ν such that $\mathfrak{A} \models \Sigma$ but $\mathfrak{A}, \nu \not\models \varphi \approx \top$. Hence, $\nu(\neg\varphi) \neq \bot$, and so, since $\mathfrak{A}$ is permuated, there is a $d \in X$ such that $d \leq \nu(\neg\varphi)$.

For each nominal $\mathbf{j} \in \mathsf{NOM}$, let $\nu(\mathbf{j}) = s_{\mathbf{j}}$. Now, let $Z = \{s_{\mathbf{j}} \mid \mathbf{j} \in \mathsf{NOM}(\varphi)\}$, and define the following binary relation on Z: $s_{\mathbf{j}} \precsim s_{\mathbf{k}}$ iff $s_{\mathbf{j}} \leq \Diamond s_{\mathbf{k}}$. It is easy to show that $\precsim$ is a pre-order on Z. Let $d_0^1, d_0^2, \ldots, d_0^m$ be representatives from the clusters minimal with respect to $\precsim$.

Now consider the canonical extension $\mathfrak{A}^\sigma$ of $\mathfrak{A}$. Note:

1. Since all axioms except $(.3^{-1})$ of $\mathbf{LP}$ are Sahlqvist, it follows from the canonicity of Sahlqvist equations that the validity of these axioms is preserved in passing from $\mathfrak{A}$ to $\mathfrak{A}^\sigma$.

2. The validity of the equations in $\Sigma^{\approx}$ as well as $@_{\mathbf{i}}\Diamond\mathbf{j} \wedge @_{\mathbf{k}}\Diamond\mathbf{j} \leq @_{\mathbf{i}}\Diamond\mathbf{k} \vee @_{\mathbf{k}}\Diamond\mathbf{i}$ is not necessarily preserved in passing from $\mathfrak{A}$ to $\mathfrak{A}^\sigma$.

3. All the atoms in $\mathfrak{A}$ are also atoms of $\mathfrak{A}^\sigma$.

In $\mathfrak{A}^\sigma$ we have that $\Box$ is completely $\wedge$-preserving and $\Diamond$ is completely $\vee$-preserving. Thus let $\Diamond^{-1}$ denote the left-adjoint of $\Box$ in $\mathfrak{A}^\sigma$ and let $\Box^{-1}$ denote the right-adjoint of $\Diamond$ in $\mathfrak{A}^\sigma$.

Now, let $d = d_0^0$, and let $d_0^1, d_0^2, \ldots, d_0^m$ be as defined above. For each $1 \leq i \leq m$, define $D_i = \Diamond^{-1} d_0^i$, and let

$$D = \bigvee_{1 \leq i \leq m} D_i.$$

Let $X_D = \{x \in X \mid x \leq D\}$ and $\mathbf{A}_D = (A_D, \wedge^D, \vee^D, \neg^D, \bot^D, \top^D, \Diamond^D, @^D)$, where $A_D = \{a \wedge D \mid a \in A\}$, $\wedge^D$ and $\vee^D$ are the restriction of $\wedge$ and $\vee$ to A_D, and

$$\neg^D a = \neg a \wedge D \qquad \Diamond^D a = \Diamond a \wedge D$$
$$@_b^D a = @_b a \wedge D \text{ for } b \in X_D \qquad \bot^D = \bot$$
$$\top^d = D$$

Finally, let $\mathfrak{A}_D = (\mathbf{A}_D, X_D)$.

The following results can then be proved:

1. A_D is closed under the operations $\wedge^D$, $\vee^D$, $\neg^D$, $\Diamond^D$, and $@^D$.

2. $\mathfrak{A}_D$ is permeated.

3. $D_i \wedge D_j = \bot$ for $i \neq j$.

4. The mapping h: $\mathfrak{A} \to \mathfrak{A}_D$ defined by $h(a) = a \wedge D$ is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_D$, and $h|_{X_D}$: $X_D \to X_D$ is onto, and hence, $\mathfrak{A}_D \models \mathbf{LP}^{+}\Sigma^{\approx}$ (by a simple adaption of the proof of the result in universal algebra that validity is preserved under homomorphic images for our permeated algebras).

5. $\mathfrak{A}_D, \nu_D \not\models \varphi \approx \top$, where ν_d: $\mathsf{PROP} \to \mathbf{A}_d$ is defined by $\nu_d(p) = h(\nu(p))$ and ν_d: $\mathsf{NOM} \to X_d$ by $\nu_d(\mathbf{j}) = b$ (b is the interpretation of the nominal $\mathbf{j}$ in $\mathfrak{A}_d$).

6. For each $1 \leq i \leq m$, if $a, b \in A_D$ such that $a, b \neq \bot$ and $a, b \leq D_i$, then $\Diamond^D a \wedge \Diamond^D b \neq \bot$ (i.e. $\mathfrak{A}_D$ is well-connected in pieces).

Now, let S be the set of elements of $\mathfrak{A}_D$ used in the evaluation of φ and $\top$ under ν_D together with $\{D_i \mid 1 \leq i \leq m\} \cup Z \cup \{\Diamond^D z \mid z \in Z\}$, and define $\mathbf{B}_S$ as the boolean subalgebra of $\mathfrak{A}_D$ generated by S. Since S is a finite subset of A_D, $\mathbf{B}_S$ is finite. Also, $\mathbf{B}_S$ clearly preserves all boolean operations. Further, define

$$\forall x \forall y \in At\mathbf{B}_S(xRy \iff \Diamond^D x \leq \Diamond^D y),$$

and let

$$\Diamond^{\mathbf{B}} b = \bigvee \{x \in At\mathbf{B}_S \mid y \leq b \text{ and } xRy\}.$$

Consider the structure $\mathfrak{B} = (\mathbf{B}_S, \Diamond^{\mathbf{B}}, @^{\mathbf{B}}, X_{\mathbf{B}})$, where $X_{\mathbf{B}} = Z$ and

$$@_a^{\mathbf{B}} b = \begin{cases} \top & \text{if } a \leq b \\ \bot & \text{otherwise} \end{cases}$$

for $a \in X_{\mathbf{B}}$.

It follows from results in [4] that $\Diamond^{\mathbf{B}}$ is a normal operator extending $\Diamond^d$, and hence that $\mathfrak{B} \not\models \varphi \approx \top$.

To show that $\mathfrak{B} \models \mathbf{LP}\Sigma^{\approx}$ it is enough to embed $\mathfrak{B}$ in $\mathfrak{A}_d$. By modifying Bull's embedding in [3] somewhat this can indeed be done. It is in the proof we crucially use the fact that $\mathfrak{A}_d$ is well-connected in pieces.

References

[1] P. Blackburn. Representation, Reasoning, and Relational Structures: a Hybrid Logic Manifesto. *Logic Journal of the IGPL*, 8:339-365, 2000.

[2] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, Cambridge, UK, 2001.

[3] R. A. Bull. That all normal extensions of S4.3 have the finite model property. *Zeitschr. f. math. Logik und Grundlagen d. Math.* 12: 341-344, 1966.

[4] W. Conradie, W. Morton and C. van Alten. An Algebraic Look at Filtrations in Modal Logic. To appear in *Logic Journal of the IGPL.*

[5] T. Litak. Algebraization of hybrid logic with binders. In *Proceedings of RelMiCS/AKA 9*, pages 281-295, 2006.